Winter Spaces and the Stabilizer Formalism

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Background

Quantum systems can be modeled as Hilbert spaces. A quantum state can be described as density matrix $\rho \in \mathbb{B}(\mathcal{H})$, which is a positive semidefinite linear operator on the Hilbert space $\mathcal{H}$.
Evolution of quantum systems are modeled via quantum channels. That can be understood as the noise introduced to the system. A quantum channel $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ is a completely positive and state preserving map sending density matrices to density matrices.
Given any channel $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$, the Stinespring dilation theorem guarantees the existence of a Kraus representation

$$\Phi(\rho) = \sum_{a=1}^{n} E_a \rho E_a^\dagger.$$

Kraus representation is unique up to a unitary matrix.

Knill-Laflamme Subspace Condition

Given a noise model (channel) $\mathcal{E}$, to do quantum error correction, we aim to find a recovery operation $\mathcal{R}$ such that given any state in an appropriate subspace,

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho.$$

However, not all errors are correctable. For example, the error of deleting all information is nor a reversible operation. The following theorem gives a necessary and sufficient condition for all correctable errors.

Theorem (Knill-Laflamme Subspace Condition)**.** *Given a channel $\mathcal{E} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ and its Kraus operators $\{E_a\}_{a\in\Lambda}$. $\mathcal{E}$ is correctable (the recovery map exists) if and only if there exists a subspace $\mathcal{C} \subseteq \mathcal{H}$ and an orthogonal projection $P : \mathcal{H} \to \mathcal{C}$ such that*

$$PE_a^\dagger E_b P = \lambda_{ab} P.$$

This is a condition on all products of Kraus operators, which is rather inconvienient to check. Let winter graph (non-commutative graph) be defined as

$$\mathcal{V} = \operatorname{span}\left\{E_a^\dagger E_b : a, b \in \Lambda\right\}.$$

The above condition can be rewritten as

$$P\mathcal{V}P = \mathbb{C}P \quad \text{or} \quad \dim P\mathcal{V}P = 1.$$

Classical Stabilizer Formalism

The stabilizer formalism provides a concrete code space $\mathcal{C}$ for a special class of error model. Given an Abelian subgroup $G \subseteq P_n$ where P_n denote the Pauli group on n qubits. An error is correctable if and only if

$$E_a^\dagger E_b \in \operatorname{span}\{P_n \backslash N(S) \cup S\} = \operatorname{span}\{P_n \backslash \mathcal{Z}(S) \cup S\} \cong P_s,$$

where S is the stabilizer of G and $\mathcal{Z}(S)$ is the center of S.

Winter Graph Framework for Stabilizer Formalism

In operator theory, Winter graphs are exactly operator systems. That is, $\mathcal{V} \subseteq \mathbb{B}(\mathcal{H})$ where $\mathcal{V}$ is unital and self-adjoint.
A special class of Winter graph can be generated by unitary representation of the error model. Given a compact group G and a unitary representation $\pi : G \to \mathcal{U}(n)$, the Winter graph can be written as

$$\mathcal{V}_{M_0} = \operatorname{span}\{\pi(g) M_0 \pi(g) : g \in G\}.$$

To recover the stabilizer formalism, consider the stabilizer subgroup S of some abelian subgroup of Pauli group P_n. Taking the Winter graph generated by the trivial representation $\pi : S \to \mathcal{U}(n)$ where every element in S is sent to itself, it generates a class of Winter graphs

$$\mathcal{V}_{M_0} = \operatorname{span}\{g M_0 g : g \in S\}.$$

It is clear that not all M_0 give rise to an operator system. However,

$$\operatorname{span}\left\{\mathcal{V}_{M_0} : M_0 \text{ makes } \mathcal{V}_{M_0} \text{ into an operator system}\right\} = \operatorname{span}\left\{P_n \backslash \mathcal{Z}(S) \cup I^{\otimes n}\right\}$$

recovers the stabilzer formalism.

Operator Quantum Error Correction (OQEC)

A natural generalization of the stablizer formalism is the OQEC framework. Consider a decomposition of the Hilbert space

$$\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \oplus \mathcal{K}.$$

$\mathcal{H}^A$ is called a noiseless subsystem if for any $\sigma^A \in \mathcal{H}^A$ and $\sigma^B \in \mathcal{H}^B$,

$$\mathcal{E}(\sigma^A \otimes \sigma^B) = \sigma^A \otimes \tau^B$$

for some $\tau^B \in \mathcal{H}^B$.

Theorem. *$\mathcal{H}^A$ is a noiseless subsystem if and only if there exists a projection $P : \mathcal{H} \to \mathcal{H}^A \otimes \mathcal{H}^B$ such that*

$$PE_a^\dagger E_b P = (I \otimes B_{ab}) P$$

Stabilizer Formalism for OQEC

Let $S = \langle Z_1, \dots Z_s \rangle$ be the stabilizer and $N(S) = \langle i, Z_1, \dots, Z_s, X_{s+1}, Z_{s+1}, \dots, X_n, Z_n \rangle$ be its normalizer. Consider the gauge group $\mathcal{G} = \langle i, Z_1, \dots, Z_s, X_{s+1}, Z_{s+1}, \dots, X_{s+r}, Z_{s+r} \rangle$. We say this system have s stabilizer qubits, r gauge qubits, and $k = n - s - r$ logical qubits that absorb errors.
The stabilizer formalism for OQEC take advantage of the gauge qubits on the noiseless A-system and get the code space

$$E_a^\dagger E_b \in \operatorname{span}\{P_n \backslash \mathcal{Z}(S) \cup \mathcal{G}\} \cong P_{s+r}.$$

This exactly corresponds to the fact that A system is noiseless and thus all the stabilizer and gauge qubits are saved.

A Winter Space Approach in OQEC

We attempt to recover the stabilizer qubits and the gauge qubits as much as possible. Hence, we will discuss two representations that are naturally induced from the previous calculation.
Case 1: Consider $\pi : \mathcal{G} \to \mathcal{U}(n)$ where $\pi(g_s \otimes g_r) = g_s \otimes I^{\otimes r}$. It means that we will send the elements acting on the stabilizer qubits to itself, and send elements acting on the gauge qubit to identity. The code space we are able to recover is

$$\operatorname{span}\left\{\mathcal{V}_{M_0} : M_0 \text{ makes } \mathcal{V}_{M_0} \text{ into an operator system}\right\} = \operatorname{span}\left\{(P_s \backslash \mathcal{Z}_s(S)) \otimes P_r \cup I^{\otimes n}\right\}.$$

This framework recovers all the gauge qubits but only half of the stabilizer qubits.
Case 2: Hence, we will try a second representation. Consider the trivial representation $\pi : \mathcal{G} \to \mathcal{U}(n)$ where $\pi(g_s \otimes g_r) = g_s \otimes g_r$. Via this framework, we are able to get

$$\operatorname{span}\left\{\mathcal{V}_{M_0} : M_0 \text{ makes } \mathcal{V}_{M_0} \text{ into an operator system}\right\} = \operatorname{span}\left\{P_n \backslash \mathcal{Z}(\mathcal{G}) \cup I^{\otimes n}\right\}.$$

Further Generalization and Future Works

We will generalize the Winter graph framework to the following error model:
Heisenberg-Weyl Group: Generalization of Pauli group for qudits. We expect to recover a version of stabilizer codes for qudits.
Clifford Code: generalization of stabilizer codes for non-commutative error models. We will try the error model with index $\mathbb{Z}_2 \times D_8$ first.
A Winter space approach to stabilizer formalism for operator algebra quantum error correction: Generalization of our framework to the regime of operator algebra quantum error correction recently introduced by Kribs et al.

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