Monoidal Categories for Quantum Theory

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Axioms of Quantum Information

Quantum information theory follows four major postulates or axioms.

Axiom (State Space). *Any physical quantum system Q can be represented by a Hilbert space $\mathcal{H}_Q$.*

Axiom (Unitary Evolution). *Closed evolution over time of a quantum system Q can be represented by a unitary ($U^\dagger U = UU^\dagger = \mathrm{id}_{\mathcal{H}_Q}$) operator on $\mathcal{H}_Q$.*

Axiom (Multiple Systems). *Two quantum systems Q_1 and Q_2 can be considered as a joint system Q_1Q_2. The associated Hilbert space should the tensor product:*

$$\mathcal{H}_{Q_1Q_2} = \mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2}.$$

Axiom (Measurement). *Measurement of a quantum system Q corresponds to orthonormal bases of the Hilbert space $\mathcal{H}_Q$. The corresponding probabilities of measurement should follow Born's rule.*

A category is a choice of mathematical "setting" defined with objects and morphisms/processes.

Example. *The category* Hilb *has objects that are Hilbert spaces and morphisms that are (bounded) linear transformations (a linear transformation is a function $T : \mathcal{H} \to \mathcal{K}$ such that $T(v+w) = T(v)+T(w)$ and $cT(v) = T(cv)$ for any scalar $c \in \mathbb{C}$).*

Example. *The category* Quant *should have physical systems as objects and physical processes as morphisms. We can choose many different models of* Quant!

Definition (Functor). *A functor is a structure-preserving map $f : \mathcal{C} \to \mathcal{D}$ between two categories. This inherently means that there are two associated properties with the data of a functor:* (1) *an object $f(x) \in \mathcal{D}$ for every object $x \in \mathcal{C}$,* (2) *a morphism $f(x) \xrightarrow{f(\alpha)} f(y)$ in $\mathcal{D}$ for every morphism $x \xrightarrow{f} y$ in $\mathcal{C}$. Firstly, a functor respects the composition property* ($f(\alpha\beta) = f(\alpha)f(\beta)$). *Secondly, it also preserves the identity between categories* ($\mathrm{id}_{f(x)} = f(\mathrm{id}_x)$).

Since a functor "changes settings," somehow our axioms should be encoded in a functor from Quant to Hilb. But until we add more structure, we only have the State Space axioms working!

(Symmetric) Monoidal Categories

To model the Multiple Systems axiom using a functor, we need a good notion of tensor product.

Definition (Monoidal Category). *A monoidal category includes a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and a unit object $\mathbf{1} \in \mathcal{C}$ such that $\otimes$ is associative and unital with $\mathbf{1}$ up to coherent natural isomorphisms.*

For example, Quant is monoidal by considering joint systems (the unit is the empty system) and Hilb is monoidal via the usual tensor product $\otimes$ (the unit is $\mathbb{C}$).

Definition (State). *A state on an object $a \in \mathcal{C}$ is a morphism $\mathbf{1} \to a$.*

This is the unit object morphing into another object in the category, so in the case of Hilbert spaces, it "picks" out a vector.

Example. *States on $\mathcal{H}$ in* Hilb *are the linear maps $\mathbb{C} \to \mathcal{H}$, i.e., vectors in $\mathcal{H}$.*

Example. *States on a quantum system Q in* Quant *correspond to the creation operators $\varnothing \to Q$.*

Definition (Effect). *An effect on an object $a \in \mathcal{C}$ is a morphism $a \to \mathbf{1}$.*

Definition (Braiding, Symmetry). *A braiding on a monoidal category is a natural isomorphism that twists an object $a \otimes b$ into $b \otimes a$, satisfying a coherence diagram. A braiding is called a symmetry if twisting twice gets us back to $a \otimes b$.*

Both Quant and Hilb have symmetries γ and σ, so they are symmetric monoidal categories. We call a functor that preserves the structure of such categories a symmetric monoidal functor. Thus, a quantum theory satisfying both the State Space axiom and the Multiple Systems axiom is a symmetric monoidal functor

$$\mathcal{Z} : (\mathsf{Quant}, \otimes, \mathbf{1}, \gamma) \to (\mathsf{Hilb}, \otimes, \mathbf{1}, \sigma).$$

Since $\mathcal{Z}$ takes states to states, this means states of a quantum system Q are modeled by vectors in the Hilbert space $\mathcal{Z}(Q)$.

Example. *If we choose* Quant = Bord$_2$, *then the result is called a 2D topological quantum field theory.*

Dagger Categories

A dagger $\dagger : \mathcal{C} \to \mathcal{C}$ is special "functor" that meets the following properties.

1. Given any morphism $f : A \to B$, $A, B \in \mathcal{C}$, the dagger of f, or $f^\dagger$, is a map from B to A. In addition, daggers also reverse the order of composition: $(g \circ h)^\dagger = h^\dagger \circ g^\dagger$. This property is called *contravariance*
2. Given any morphism f in $\mathcal{C}$, $f^{\dagger\dagger} = f$. This property is called *involutivity*
3. The dagger is the identity morphism on objects.

Definition (Dagger Category). *A category equipped with a dagger $(\mathcal{C}, \dagger)$ is called a dagger category.*

We define two types of morphisms that behave nicely with daggers.

1. *Unitary morphisms.* A unitary morphism $f : A \to B$ has the property that $f \circ f^\dagger = \mathrm{id}_B$ and $f^\dagger \circ f = \mathrm{id}_A$.
   Note. Unitary functions are useful because they allow us to *reverse* any transformation we apply to our objects. From a quantum perspective, it means we can reverse our evolutions!
2. *Isometry.* An isometry f only has the property that $f^\dagger f = \mathrm{id}_A$ (like half a unitary).
   Note. These types of morphisms will be used when incorporating axiom 4.

Definition (Dagger Monoidal Category). *A dagger monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \rho, \lambda, \dagger)$ is a monoidal category with a dagger operation where the coherence natural isomorphisms α, ρ, and λ are unitary, and $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ (f, g are morphisms).*

Note. If we also add a braiding/symmetry γ to our category, the only extra condition is for γ to be unitary! In this case, we add the adjective braided/symmetric.

Example. *Both* Hilb *and* Quant *are dagger symmetric monoidal categories. In* Hilb, *we use the adjoint as the $\dagger$. In* Quant, *we reverse our closed evolution to take the $\dagger$.*

Dagger functors are functors that preserve the dagger. This notion can be combined with that of a symmetric monoidal functor. A quantum theory satisfying the State Space axiom, the Unitary Evolution axiom, and the Multiple Systems axiom is then a dagger symmetric monoidal functor Quant $\to$ Hilb.

Enrichment and Biproducts

Definition (Zero Object). *A zero object in a category $\mathcal{C}$ is an object $0 \in \mathcal{C}$ such that $\forall a \in \mathcal{C}$, $\exists!(a \to 0)$ and $\exists!(0 \to a)$.*

Definition (Zero Morphism). *Let $(\mathcal{C}, 0)$ be a category with a zero object. Then a composite $A \to 0 \to B$ in $\mathcal{C}$ exists and is uniquely determined. We call it the zero morphism $0_{A,B}$.*

We say a category with a zero object $(\mathcal{C}, 0)$ is enriched in commutative monoids (CMon) if its morphisms can be added (in a way that is compatible with composition) associatively so that the zero morphisms act as units.

Definition (Biproduct). *Let $(\mathcal{C}, 0)$ be a category with a zero object that is enriched in* CMon. *Then, given $a, b \in \mathcal{C}$, their biproduct is an object $a \oplus b \in \mathcal{C}$ with morphisms:*

- $i_a : a \to a \oplus b$ *and* $p_a : a \oplus b \to a$.
- $i_b : b \to a \oplus b$ *and* $p_b : a \oplus b \to b$.

Such that:

- $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_a} a = \mathrm{id}_a$ *and* $b \xrightarrow{i_b} a \oplus b \xrightarrow{p_b} b = \mathrm{id}_b$.
- $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_b} = 0_{a,b}$ *and* $b \xrightarrow{i_b} a \oplus b \xrightarrow{p_a} = 0_{b,a}$.
- $i_1p_1 + i_2p_2 = \mathrm{id}_{a\oplus b}$.

Biproducts give us superposition. Note that a dagger biproduct is a biproduct in a dagger category where the $i_a^\dagger = p_a$ and $i_b^\dagger = p_b$.

Definition (Probability). *Given a state s and an effect e in a dagger monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \dagger)$, the probability Prob$(s$ in $e)$ is given by $s^\dagger \circ e^\dagger \circ e \circ s$ or*

$$\mathbf{1} \xrightarrow{s} a \xrightarrow{e} \mathbf{1} \xrightarrow{e^\dagger} a \xrightarrow{s^\dagger} \mathbf{1}.$$

Definition (Complete Set). *A set of effects $\{a \xrightarrow{e_\lambda} \mathbf{1}\}_{\lambda\in\Lambda}$ is complete if for any non-zero morphism $b \xrightarrow{f} a$ (ie $f \neq 0_{b,a}$) there is an effect $e_{\lambda'}$, $\lambda' \in \Lambda$ such that $e_{\lambda'} \circ f \neq 0_{b,\mathbf{1}}$.*

Using these two notions, we can now state Born's Rule.

Theorem (Born's Rule). *If a set of effects $\{e_1, e_2, \ldots, e_n\}$, $n \in \mathbb{N}$, is complete in a dagger monoidal category with a zero object and dagger biproducts $(\mathcal{C}, \otimes, \mathbf{1}, \dagger, 0, \oplus)$, then for any isometry $x : \mathbf{1} \to a$, we have*

$$\sum_{i=1}^{n} \mathit{Prob}(x \text{ in } e_i) = 1,$$

where 1 is the identity morphism on $\mathbf{1}$.

We can quickly prove this theorem (assuming one other fact).

Proof. We can first expand and factor this sum via rules of biproduct and Prob(s in e):

$$\sum_{i=1}^{n} \mathrm{Prob}(x \text{ in } e_i) = \sum_{i=1}^{n} x^\dagger e_i^\dagger e_i x = x^\dagger \left(\sum_{i=1}^{n} e_i^\dagger e_i \right) x.$$

From here, we assume the fact $\sum_{i=1}^{n} e_i^\dagger e_i = \mathrm{id}_a$ (proved using biproducts) to finish the proof:

$$x^\dagger \left(\sum_{i=1}^{n} e_i^\dagger e_i \right) x = x^\dagger \mathrm{id}_a x = x^\dagger x = \mathrm{id}_{\mathbf{1}} = 1.$$

□

Thus, a quantum theory satisfying all four axioms is a dagger symmetric monoidal functor from Quant to Hilb that fully preserves the biproduct structure!

References

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