A model for the infinite tensor product of groups

Insights from quantum information

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Overview

The contemporary mathematical paradigm for quantum mechanics can be summarized via four axioms.

The axioms give a account of how a quantum system can be modeled using the rich mathematical framework of Hilbert spaces.

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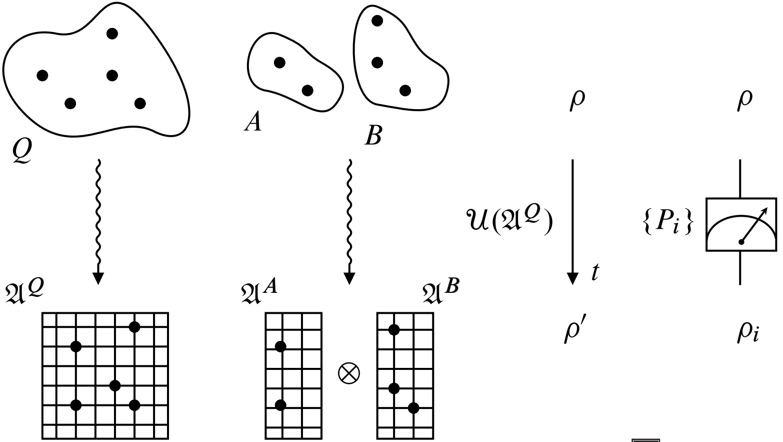
Quantum Axioms Visualized



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Pauli Group

We call Hilbert spaces $\mathfrak{A} \simeq \mathbb{C}^2$ qubits.

Pauli Group

The *Pauli group* $\mathscr{P}$ is the nonabelian matrix group generated by

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

There is a natural action of $\mathscr{P}$ on a qubit $\mathfrak{A}$.

If a qubit is "modeled" by the space $\mathbb{C}^2$, then n qubits should be modeled by the space

$$\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}} = (\mathbb{C}^2)^{\otimes n}.$$

We can generalize the Pauli group to n qubits.

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n-Qubit Pauli Group

Denote a 1-local action of $\Sigma \in \mathscr{P}$ on qubit j of $\mathfrak{A} \simeq \bigotimes_j \mathbb{C}^2$ by

$$\Sigma_j := I_2 \otimes I_2 \otimes \cdots \underbrace{\otimes \Sigma \otimes}_{j\text{th position}} \cdots \otimes I_2.$$

Then, define the n-qubit Pauli group

$$\mathscr{P}_n := \langle i I_j, X_j, Z_j : 1 \leq j \leq n \rangle$$

If $\mathfrak{A} \simeq (\mathbb{C}^2)^{\otimes n}$, then $\Sigma_j \in \mathscr{P}_n$ and $\mathscr{P}_n \curvearrowright \mathfrak{A}$.

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In the Limit

Q: What happens when we take $n \to \infty$? How could we do algebra in ∞-qubit systems?

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In the Limit

$$\begin{Bmatrix} n\text{-qubit systems} \\ \mathcal{P}_n \\ (\mathbb{C}^2)^{\otimes n} \end{Bmatrix} \rightsquigarrow \begin{Bmatrix} \infty\text{-qubit systems} \\ \mathcal{P}_\infty \\ \mathfrak{G} \end{Bmatrix}$$

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Formal Approach

There are natural inclusions

$$\mathcal{P} = \mathcal{P}_1 \xhookrightarrow{\iota_1^2} \mathcal{P}_2 \xhookrightarrow{\iota_2^2} \cdots \xhookrightarrow{\iota_{n-1}^n} \mathcal{P}_n \xhookrightarrow{\iota_n^{n+1}} \cdots$$

which send

$$\sigma \mapsto \sigma \otimes I_2.$$

Take the direct limit and define $\mathcal{P}_\infty := \operatorname{colim} \langle \mathcal{P}_n, \iota_n^m \rangle$.

Constructive Approach

Consider the invertible 2×2 matrices $\mathrm{GL}_2(\mathbb{C})$. If $A_1, \ldots, A_n$ are invertible, then we can form n-fold tensor matrices

$$A_1 \otimes A_2 \otimes \cdots \otimes A_n.$$

We call the group of all these tensors $\mathrm{GL}_2(\mathbb{C})^{\otimes n}$.

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Constructive Approach

To generalize, we could take all sequences of tensors[1]

$$A_{\otimes} := A_1 \otimes A_2 \otimes \cdots \otimes A_i \otimes \cdots$$

and form a group $\overline{\mathrm{GL}_2^{\infty}(\mathbb{C})}$. We call these $A_{\otimes}$ the $\mathbb{N}$-fold tensor map associated to $\{A_i\}_{i \in \mathbb{N}}$.

[1] In practice, it is easier to consider the A_i as automorphisms $\mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2$.

Constructive Approach

But doing some algebra tells us we want finite multilinearity. Instead, consider all finitely supported sequences and form the subgroup of restricted $\mathbb{N}$-fold tensor maps $\mathrm{GL}_2^{\infty}(\mathbb{C})$.

Here, all but finitely many of the A_i are trivial.

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Constructive Approach

We know $\mathscr{P}$ lives in $\mathrm{GL}_2(\mathbb{C})$, so take all finitely supported tensor sequences from $\mathscr{P}$ and form $\mathscr{P}_\infty$ in $\mathrm{GL}_2^\infty(\mathbb{C})$.

We call this our ∞-qubit Pauli group.[2]

[2] Finding the correct Hilbert space $\mathfrak{G}$ for $\mathscr{P}_\infty$ to act on requires some care. It turns out a natural setting is (isomorphic to) $\ell^2[\Omega]$ for a well-chosen Ω.

Error Correction

We model quantum errors as quantum channels.

(i) A *superoperator* is a linear map $\mathcal{E} : \mathbb{B}(\mathfrak{A}) \to \mathbb{B}(\mathfrak{B})$.

(ii) A *quantum channel* $\mathcal{E}$ is a superoperator which is completely positive and trace-preserving.

That is, $\mathcal{E} \otimes \mathrm{id}_k \geq 0$ for all k and $\mathrm{tr}(\mathcal{E}\rho) = \mathrm{tr}(\rho)$.

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Error Correction

A theorem due to Karl Kraus tells us that every quantum channel $\mathcal{E} : \mathbb{B}(\mathfrak{A}) \to \mathbb{B}(\mathfrak{A})$ has a decomposition[3]

$$\mathcal{E}(-) = \sum_{x \in X} E_x(-)E_x^\dagger,$$

where $\{E_x\}_{x \in X} \subseteq \mathbb{B}(\mathfrak{A})$.

[3] Tracking down precisely how large X is and in what topology convergence works can be a bit challenging. If $\mathfrak{A}$ is separable, then $X = \mathbb{N}$ allows ultraweak convergence.

Some terminology:

(i) A *codespace* is a subspace $\mathcal{C} \subseteq \mathfrak{A}$.

(ii) Given an error $\mathcal{E} : \mathbb{B}(\mathfrak{A}) \to \mathbb{B}(\mathfrak{A})$, we call $\mathcal{R} : \mathbb{B}(\mathfrak{A}) \to \mathbb{B}(\mathfrak{A})$ a *recovery channel* if for all states $\rho \in \mathbb{B}(\mathfrak{A})$,

$$(\mathcal{R} \circ \mathcal{E})(\rho) \propto \rho.$$

on the codespace.

(iii) An error $\mathcal{E}$ is *correctable* if a codespace $\mathcal{C}$ and recovery channel $\mathcal{R}$ exist.

Gottesman's Stabilizer Formalism

Theorem (Gottesman, 1997)

Let $\mathfrak{A} \simeq (\mathbb{C})^{\otimes n}$, let $\mathcal{E}$ be a noise channel on $\mathfrak{A}$ with Kraus operators $\{E_i\}_{i=1}^r$, and let $\mathcal{S} \leq \mathcal{P}_n$ be a finitely-generated abelian subgroup without $-(I^{\otimes n})$. Then, $\mathcal{E}$ is correctable on $\mathrm{Fix}_{\mathcal{S}}(\mathfrak{A})$ if and only if for all $1 \leq i, j \leq r$, we have

$$E_i^\dagger E_j \in \mathrm{span}\{(\mathcal{P}_n \setminus \mathcal{N}_{\mathcal{P}_n}(\mathcal{S})) \cup \mathcal{S}\} \subseteq \mathbb{B}(\mathfrak{A}),$$

where $\mathcal{N}_{\mathcal{P}_n}(\mathcal{S})$ is the normalizer of $\mathcal{S}$ in the n-qubit Pauli group.

∞-Qubit Stabilizer Formalism

Letting $\mathfrak{G}$ be an ∞-qubit space, can we recover the Stabilizer Formalism for errors on $\mathfrak{G}$ by simply "exchanging n with ∞?"

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