BREDON COHOMOLOGY FOR SMITH THEORY

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Abstract. In this brief expository paper, we develop the basic theory of Bredon cohomology, an equivariant analogue of ordinary cohomology. Rather immediately, we will be able to apply this machinery to prove some results about modulo-p cohomology spheres due to Paul Althaus Smith [Smi38]. We follow [May96].

In the ordinary algebraic topology of spaces, we try and study the relationship between (suitably nice) topological spaces, the continuous maps therebetween, and corresponding algebraic structures equipped with their usual notion of map. In particular, our goal is often to *distinguish* spaces, so we proceed functorially.

$$\begin{Bmatrix}\text{spaces and}\\ \text{continuous maps}\end{Bmatrix} \longrightarrow \begin{Bmatrix}\text{algebraic objects}\\ \text{and homomorphisms}\end{Bmatrix}$$

This framework leads us to define the usual invariants like the homotopy groups and singular (co)homology. The story of *equivariant* algebraic topology is analogous. After allowing spaces to have a (continuous) group action attached to them, our goal is to construct algebraic invariants that have the power to distinguish such objects.

$$\begin{tikzcd}
\begin{Bmatrix}\text{nice}\\ \text{groups}\end{Bmatrix} \arrow[dd, squiggly] & & & \\
 & = & \begin{Bmatrix}\text{spaces with an action}\\ \text{and equivariant maps}\end{Bmatrix} \arrow[r] & \begin{Bmatrix}\text{algebraic objects}\\ \text{and homomorphisms}\end{Bmatrix} \\
\begin{Bmatrix}\text{spaces and}\\ \text{continuous maps}\end{Bmatrix} & & &
\end{tikzcd}$$

We will begin here in §1, defining the basic structures, notably G-spaces and G-CW complexes, necessary to build the Bredon equivariant cohomology in §2. Then, in §3, we will apply our equivariant theory to prove the following result for a prime p.

Theorem 0.1 (P. A. Smith, 1938)**.** *Let G be a finite p-group and X be a finite-dimensional G-CW complex that is a modulo-p cohomology n-sphere. Then, there is an $m \leq n$ so that X^G is either empty or a modulo-p cohomology m-sphere. If $p \neq 2$, then $n - m$ is even, and even n means X^G is nonempty.*

We will denote by $\mathcal{S}\mathrm{et}$ the category of sets, by $\mathfrak{A}\mathrm{b}$ the category of abelian groups, by $\mathcal{V}\mathrm{ect}(\Bbbk)$ the category of $\Bbbk$-vector spaces, by $\mathcal{T}\mathrm{op}$ the category of (nice, i.e., CGWH) spaces, and for categories $\mathcal{C}$ and $\mathfrak{D}$, by $\mathfrak{F}\mathrm{un}(\mathcal{C}, \mathfrak{D})$ the functor category between them.

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1. Spaces with Actions

Let G be a *topological group*, i.e., a group object in $\mathcal{T}\mathrm{op}$. In the way that a G-set is nothing more than a $\mathcal{S}\mathrm{et}$-valued functor from G and a G-representation is nothing more than a $\mathcal{V}\mathrm{ect}(\Bbbk)$-valued functor from G, a G-space should assign to each element of G a homeomorphism of the space X, such that this assignment is compatible with the group structure.

Definition 1.1 (G-Space)**.** A G-space is a topological space X equipped with a continuous group action $G \times X \to X$.

In analogy with the theories of G-sets or G-representations, we need a way to talk about maps on a G-space X that respect the action. Precisely, we will say a *G-equivariant map* $f : X \to Y$ between two G-spaces is a continuous map such that for all $g \in G$ and $x \in X$,

$$f(gx) = gf(x).$$

That is, we need the naturality square

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

to commute. Note that G-equivariant maps are also called G-maps.

Definition 1.2 (Category of G-Spaces)**.** Let $\mathcal{T}\mathrm{op}(G)$ denote the category whose objects are G-spaces and whose morphisms are G-equivariant maps.

The usual constructions from a G-set translate over to the setting of G-spaces.

Definition 1.3 (Stabilizer, Orbit, Fixed Point Space)**.** Let X be a G-space.

(i) The stabilizer of a point $x \in X$ is the subgroup $G_x = \{g \in G : gx = x\} \subseteq G$. If X is reasonably nice (T_1), then G_x is a closed subgroup.
(ii) The orbit of a point $x \in X$ is the subspace $x^G = \{gx \in X : g \in G\} \subseteq X$.
(iii) The fixed point space X^H of a closed subgroup $H \subseteq G$ is the space of all $x \in X$ such that $hx = x$ for all $h \in H$.

If $H \subseteq G$ is a closed subgroup, we will write G/H for the *coset space*. Coset spaces G/H naturally correspond to orbits x^G for $x \in G$, so we will often call the spaces G/H the orbits. The following slogan in [May96] helps motivate much of the theory.

> *"In equivariant theory, orbits G/H play the role of points, and the set of G-maps $G/H \to G/H$ can be identified with the group WH."*

Here, WH is just the quotient of the normalizer of H in G by H.

Definition 1.4 (Orbit Category)**.** Given a G-space X, define the orbit category $\mathcal{O}_G \subseteq \mathcal{T}\mathrm{op}(G)$ to be the full subcategory cut out by the orbits G/H.

In practice, the usual notions of isomorphism in $\mathcal{T}\mathrm{op}(G)$ and $\mathcal{O}_G$ are too strong, and we will want to work with our new equivariant structures in a homotopical setting. If $f, g : X \rightrightarrows Y$ are G-equivariant maps, we say that $H : X \times I \to Y$ is a *homotopy* from f to g if H is a G-equivariant homotopy in the usual sense, where we let G act trivially on the interval. Then, a *homotopy equivalence* is a G-equivariant map that is an isomorphism of G-spaces up to homotopy.

Definition 1.5 (Homotopy Category)**.** Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a wide subcategory. The homotopy category of $\mathcal{C}$ localized at $\mathcal{W}$ is a category $\mathcal{C}[\mathcal{W}^{-1}]$ with a functor

$$\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[\mathcal{W}^{-1}]$$

such that

(i) if $f \in \mathrm{mor}(\mathcal{W})$, then $\gamma(f)$ is an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$.
(ii) if $\tau : \mathcal{C} \to \mathfrak{D}$ is any functor so that $\tau(\mathcal{W})$ are all $\mathfrak{D}$-isomorphisms, then there is a unique functor

$$\widetilde{\tau} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathfrak{D}$$

such that the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\tau"] \arrow[d, "\gamma"'] & \mathfrak{D} \\ \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, dashed, "\widetilde{\tau}"'] & \end{tikzcd}$$

commutes.

That is, the homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ is a sort of *category of fractions*, where we have formally inverted the morphisms in $\mathcal{W}$.

Example 1.6 (Naïve Homotopy Category)**.** Let $\mathcal{C} = \mathcal{T}\mathrm{op}$ and $\mathcal{H}$ be the wide subcategory of homotopy equivalences. Then, $\mathcal{C}[\mathcal{H}^{-1}]$ is the usual (naïve) homotopy category one encounters in a first course in algebraic topology, written $h\mathcal{T}\mathrm{op}$.

Example 1.7 (CW Complexes)**.** When we actually do algebraic topology, we tend to restrict ourselves to the subcategory $\mathcal{C}\mathrm{W} \subseteq \mathcal{T}\mathrm{op}$ of CW complexes and cellular maps. Inverting homotopy equivalences, we obtain a full subcategory $h\mathcal{C}\mathrm{W} \subseteq h\mathcal{T}\mathrm{op}$.

Example 1.8 (Orbits and G-Spaces)**.** We will write $h\mathcal{O}_G$ for the homotopy category $\mathcal{O}_G[\mathcal{H}^{-1}]$, where $\mathcal{H}$ is homotopy equivalences in the equivariant sense. Likewise, we will write $h\mathcal{T}\mathrm{op}(G)$ for the homotopy category of G-spaces localized at the homotopy equivalences.

In many settings, however, we can get away with a notion even weaker than homotopy. In the ordinary theory, we say a map $f : X \to Y$ of spaces is a weak equivalence if it induces an isomorphism $f_* : \pi_n(X) \to \pi_n(Y)$ for all $n \in \mathbb{N}$. Switching to the equivariant picture, we will say that a G-equivariant map $f : X \to Y$ of G-spaces is a *weak equivalence* if the induced map $f^H : X^H \to Y^H$ on the fixed point spaces of any closed $H \subseteq G$ is a weak equivalence in the ordinary sense.

Example 1.9 (True Homotopy Category)**.** We could have instead defined a homotopy category of spaces to be $\mathcal{T}\mathrm{op}[\mathcal{W}^{-1}]$, where $\mathcal{W}$ is the wide subcategory of spaces and weak equivalences.

Example 1.10 (Weak G-Spaces)**.** Let $w\mathcal{T}\mathrm{op}(G)$ denote the homotopy category of G-spaces localized at the equivariant weak equivalences.

A standard result is that the homotopy theory of CW complexes is the same as the weak homotopy theory of spaces. That is, there is an equivalence of categories

$$\mathcal{T}\mathrm{op}[\mathcal{W}^{-1}] \simeq h\mathcal{C}\mathrm{W} \subseteq h\mathcal{T}\mathrm{op}.$$

Our ordinary invariants are then defined on the CW homotopy category. If our goal is to extend such theories to G-spaces, we should thus have a sturdy notion of a G-CW complex. By analogy, this should be a G-space that has the G-action integrated into each cell of a CW complex in a coherent way. Moreover, it should have the same homotopy theory as the weak homotopy theory of G-spaces.

Definition 1.11 (G-CW Complex)**.** A G-CW complex X is a G-space with n-skeleton X^n defined recursively in the following way:

(i) the 0-skeleton X^0 is given by a disjoint union of orbits G/H.

(ii) by picking G-equivariant maps

$$G/H \times \mathbb{S}^n \to X^n,$$

we can form the pushout

$$X^{n+1} = \operatorname{colim}\left(\coprod_H G/H \times \mathbb{D}^{n+1} \hookleftarrow \coprod_H G/H \times \mathbb{S}^n \to X^n\right),$$

where G acts trivially on the disks and spheres.

(iii) X is the directed colimit of the n-skeleta with the inclusions $X^n \to X^{n+1}$.

We now have a category $\mathcal{CW}(G) \subseteq \mathcal{T}\mathrm{op}(G)$ of G-CW complexes and cellular maps. Of course, we could have instead defined relative G-CW complexes (X, A), in which case we specify that the 0-skeleton also has a copy of A. Using this definition of G-CW complex, it can be shown that the equivariant theory satisfies the expected versions of the homotopy extension and lifting property, Whitehead's theorem, cellular approximation, and (G-)CW approximation. See [May96] for proofs. These results are enough to conclude that there is an equivalence of categories

$$w\mathcal{T}\mathrm{op}(G) \simeq h\mathcal{CW}(G) \subseteq h\mathcal{T}\mathrm{op}(G),$$

where $h\mathcal{CW}(G)$ is the homotopy category of $\mathcal{CW}(G)$ localized at the equivariant homotopy equivalences. (Of course, we could again consider pairs, yielding the category $h\mathcal{CW}_2(G)$.) That is, the weak homotopy theory of G-spaces is equivalent to the homotopy theory of G-CW complexes, so we will define our invariant, Bredon cohomology, on the homotopy category of G-CW complexes.

2. Bredon Cohomology

To define a cohomology theory in the equivariant setting, we should first determine what category our coefficients lie in. We will call any functor $h\mathcal{O}_G^{\mathrm{op}} \to \mathfrak{Ab}$ a *coefficient system*. That is, coefficient systems are $\mathfrak{Ab}$-valued presheaves on the homotopy category of orbits. Evidently, this is contravariant, since we have taken the opposite homotopy category of orbits in the domain. If our goal was to define an equivariant homology theory, we would instead use covariant coefficient systems $h\mathcal{O}_G \to \mathfrak{Ab}$. We can arrange coefficient systems into a category $\mathfrak{Coef}$ by letting morphisms be natural transformations. That is, define the category of coefficient systems

$$\mathfrak{Coef} = \mathfrak{Fun}(h\mathcal{O}_G^{\mathrm{op}}, \mathfrak{Ab}).$$

The following is a standard result. For a proof, see Proposition 5.93 of [Rot09].

Theorem 2.1. *If $\mathfrak{D}$ is an abelian category, then so is $\mathfrak{Fun}(\mathcal{C}, \mathfrak{D})$.*

In particular, this means our coefficient systems form an abelian category, so we can do homological algebra with them, i.e., we can define and manipulate chain complexes of coefficient systems. We will write $\mathfrak{Ch}_\bullet(\mathfrak{Coef})$ (resp. $\mathrm{co}\mathfrak{Ch}_\bullet(\mathfrak{Coef})$) for the category of (resp. co)chain complexes in coefficient systems with chain maps.

Let $C : h\mathcal{O}_G^{\mathrm{op}} \to \mathfrak{Ab}$ be a coefficient system. By a (relative, unreduced) *Bredon equivariant cohomology theory*, we will mean then a sequence of functors

$$H_G^\bullet(-,-;C) : h\mathfrak{CW}_2(G) \to \mathfrak{Ab},$$

along with natural transformations δ that have components

$$\delta_{(X,A)} : H_G^\bullet(X,\varnothing;C) \to H_G^{\bullet+1}(X,A;C)$$

such that this data satisfies equivariant analogues of the Eilenberg-Steenrod axioms for ordinary unreduced cohomology. Note that while the rest of the axioms look like their ordinary counterparts, the dimension axiom should be treated with care. In particular, we follow the slogan that orbits are points, so that for each orbit G/H,

$$H_G^n(G/H,\varnothing;C) = \begin{cases} C(G/H), & n = 0 \\ 0, & \text{otherwise}, \end{cases}$$

where $C(G/H)$ is the evaluation of our coefficient system on the orbit space. As we would expect, such a theory, if one exists, is unique.

Theorem 2.2. *Let C be a coefficient system. Then, there is no more than one natural isomorphism class of Bredon equivariant cohomology theories $H_G^\bullet(-,-;C)$.*

The proof of uniqueness is similar to the proof for ordinary cohomology theories. We now construct cellular Bredon cohomology, which can then be shown to satisfy the Eilenberg-Steenrod axioms.

Definition 2.3 (Cellular Bredon Chain Complex)**.** Let X be a G-CW complex. Then, the cellular Bredon chain complex $\underline{C}_\bullet \in \mathfrak{Ch}_\bullet(\mathfrak{Coef})$ assigns to each n and each orbit G/H the relative integral homology (in ordinary spaces) of the pair of fixed point spaces $((X^n)^H, (X^{n-1})^H)$. That is,

$$\underline{C}_n(G/H) = H_n((X^n)^H, (X^{n-1})^H; \mathbb{Z}).$$

In ordinary spaces, there is a connecting homomorphism associated to the triple

$$((X^n)^H, (X^{n-1})^H, (X^{n-2})^H),$$

so we obtain the desired differential

$$d : \underline{C}_n(X) \to \underline{C}_{n-1}(X).$$

Definition 2.4 (Cellular Bredon Cohomology)**.** Let X be a G-CW complex and C be a coefficient system. Then, the cellular Bredon cohomology of X is the cohomology

$$H_{G,\mathrm{cell}}^\bullet(X;C) = H^\bullet(\mathfrak{Coef}(\underline{C}_\bullet(X), C)).$$

That is, we take the cohomology of the cochain complex obtained by considering maps from the cellular Bredon complex into our chosen coefficient system C.

We have thus constructed a sequence of functors

$$H_{G,\mathrm{cell}}^\bullet(-;C) : \mathfrak{CW}(G) \xrightarrow{\underline{C}_\bullet} \mathfrak{Ch}_\bullet(\mathfrak{Coef}) \xrightarrow{\mathfrak{Coef}(-,C)} \mathrm{co}\mathfrak{Ch}_\bullet(\mathfrak{Coef}) \xrightarrow{H^\bullet} \mathfrak{Ab}.$$

Note that homotopic G-equivariant maps induce isomorphisms on cellular Bredon cohomology, so $H_{G,\mathrm{cell}}^\bullet(-;C)$ descends to a functor from $h\mathfrak{CW}(G) \simeq w\mathfrak{Top}(G)$.

The remaining axioms can be shown to be satisfied by this cellular construction. Hereafter, by *Bredon cohomology* we mean any theory isomorphic to cellular Bredon cohomology, writing $H^\bullet_G(-;C)$. For ordinary homology with coefficients in $\mathbb{F}_p$, we write $H^\bullet(-) : h\mathcal{C}\mathrm{W} \to \mathcal{V}\mathrm{ect}(\mathbb{F}_p)$, using the notation $\widetilde{H}^\bullet(-)$ for the reduced version.

3. Smith Theory

We now turn our attention back to proving Theorem 0.1. The intent is to use our new equivariant tool Bredon cohomology, plus some rudimentary algebra, to prove Smith's results about ordinary cohomology spheres. We begin with the following simple observation about fixed point subspace.

Lemma 3.1. *Let $H \lhd G$ be a nontrivial normal subgroup. Then, $X^G = (X^H)^{G/H}$, where $X^H \subseteq X$ is interpreted as a G/H-space in the natural way.*

Proof. We compute

$$\begin{aligned} X^G &= \{x \in X : gx = x \text{ for all } g \in G\} \\ &= \{x \in X : ghx = x \text{ for all } g \in G \text{ and all } h \in H\} \\ &= \{x \in X^H : (gH)x = x \text{ for all } g \in G\}, \end{aligned}$$

which is precisely $(X^H)^{G/H}$. □

We now recall the notion of solvability for groups, which will allow us to make a notable reduction in the complexity of the problem.

Definition 3.2 (Solvable Group)**.** A group G is solvable if there is a chain

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

where G_i/G_{i-1} is abelian for $1 \leq i \leq n$.

Then, the following is a brief exericse in induction.

Exercise 3.3. *Finite p-groups are solvable.*

Corollary 3.4. *If Theorem* 0.1 *holds in the cyclic case $G = \mathbb{Z}/p$, it also holds for arbitrary finite p-groups G.*

Proof. Since G is solvable, there is some $H \lhd G$ so that $G/H \simeq \mathbb{Z}/p$. By Lemma 3.1, we can focus on X^H as a $\mathbb{Z}/p$-space. Induction tells us that either X^H is empty or a cohomology sphere of the desired type, which implies the result for X^G. □

Let us establish some notation. Per Corollary 3.4, G will be the cyclic group $\mathbb{Z}/p$, unless otherwise noted. We define FX as the G-space X_+/X^G, where X_+ is the G-space union $X \cup *$ with a disjoint basepoint. While we have not explicitly mentioned basepoints until now, the basics of our equivariant theory continue to work as expected with based G-spaces. Per usual, we refer the reader to [May96] for details. We also define three coefficient systems $L, M, N \in \mathcal{C}\mathrm{oef}$ such that

$$H^n_G(X;L) \simeq \widetilde{H}^n(FX/G), \quad H^n_G(X;M) \simeq H^n(X), \quad \text{and} \quad H^n_G(X;N) \simeq H^n(X^G).$$

That such coefficient systems exist can be shown by defining on objects

$$L(c) = \begin{cases} \mathbb{F}_p, & c = G \\ 0, & c = *, \end{cases}$$

$$M(c) = \begin{cases} \mathbb{F}_p[G], & c = G \\ \mathbb{F}_p, & c = *, \end{cases}$$

where $\mathbb{F}_p[G]$ is the group ring, and

$$N(c) = \begin{cases} 0, & c = G \\ \mathbb{F}_p, & c = *. \end{cases}$$

Observe that the uniqueness of a Bredon equivariant cohomology theory satisfying the axioms (Theorem 2.2) assures us that this is enough data to give the desired characterization. Finally, we define the following shorthands for dimensions:

$$\begin{aligned} a_n &= \dim\left(\widetilde{H}^n(FX/G)\right) \\ \overline{a}_n &= \dim\left(H^n_G(X; I)\right) \\ b_n &= \dim\left(H^q(X)\right) \\ c_n &= \dim\left(H^n(X^G)\right). \end{aligned}$$

We now need to do some algebra.

Definition 3.5 (Augmention Map, Ideal)**.** Define the augmentation map ε to be the usual ring homomorphism

$$\varepsilon : \mathbb{F}_p[G] \to \mathbb{F}_p$$

so that $\varepsilon(g) = 1$ for all $g \in G$. The augmentation ideal of $\mathbb{F}_p[G]$ is the ideal $I = \ker(\varepsilon)$.

Note that this definition makes sense for any group ring $R[G]$. See [Bro82] for characterizations and applications of the augmentation ideal in group cohomology. In particular, it can be shown that ε is surjective. Define a coefficient system I by

$$I(c) = \begin{cases} I, & c = G \\ 0, & c = *. \end{cases}$$

Lemma 3.6. *We have a pair of short exact sequences of coefficient systems*

$$0 \to I \to M \to L \oplus N \to 0$$

and

$$0 \to L \to M \to I \oplus N \to 0.$$

Proof. When we evaluate on $*$, both sequences are

$$0 \to 0 \to \mathbb{F}_p \to 0 \oplus \mathbb{F}_p \to 0,$$

which is exact. When we evalute on G, we get the sequences

$$0 \to I \to \mathbb{F}_p[G] \to \mathbb{F}_p \oplus 0 \to 0$$

and

$$0 \to \mathbb{F}_p \to \mathbb{F}_p[G] \to I \oplus 0 \to 0.$$

The rank-nullity theorem tells us that these sequences are not just exact, but split exact, since $I = \ker(\varepsilon)$ and $\mathbb{F}_p = \varepsilon(\mathbb{F}_p[G])$. $\square$

Corollary 3.7. *There are long exact sequences*

$$\cdots \to H^n_G(X; I) \to H^n(X) \to \widetilde{H}^n(FX/G) \oplus H^n(X^G) \to H^{n+1}_G(X; I) \to \cdots$$

and

$$\cdots \to \widetilde{H}^n(FX/G) \to H^n(X) \to H^n_G(X; I) \oplus H^n(X^G) \to \widetilde{H}^{n+1}(FX/G) \to \cdots.$$

Proof. We get two long exact sequences in Bredon cohomology that arise from the short exact sequences in Lemma 3.6 in the usual way. Then, just use the definitions of the coefficient systems L, M, and N to obtain the above sequences in terms of ordinary and Bredon cohomology. □

By bounding the dimensions of images, the first long exact sequence gives us the inequality $b_n + c_n \leq b_n + \overline{a}_{n+1}$. Likewise, the second long exact sequence yields $\overline{a}_n + c_n \leq b_n + a_{n+1}$. Together, these give the inequality

$$2c_n + b_n + \overline{a}_n \leq 2b_n + a_{n+1} + \overline{a}_{n+1}.$$

Summing over both sides, it is simple to observe that

$$\sum c_n \leq \sum b_n.$$

We will now need one (rather, $p-1$) more exact sequence(s).

Lemma 3.8. *For $1 \leq n \leq p-1$, there is a short exact sequence of coefficient systems*

$$0 \to I^{n+1} \to I^n \to L \to 0$$

where I^n is the nth power of the coefficient system I.

Idea. We omit the proof, but note that this falls easily from the ideal structure of I^n, which in turn, follows from the ideal structure of the augmentation ideal I. See the exercises of Chapter 1, §2 in [Bro82] for the latter. □

The above short exact sequences give rise to cohomological long exact sequences in the usual way. Filling in the standard Euler characteristic formula for each term in the sequences and summing over $1 \leq n \leq p-1$, we obtain the following.

Corollary 3.9. *The Euler characteristic of X admits*

$$\chi(X) = \chi(X^G) + p\chi(FX/G) - p.$$

In particular, $\chi(X)$ is just $\chi(X^G)$ modulo-p.

We can now prove our main result.

Theorem 0.1 (P. A. Smith, 1938)**.** *Let G be a finite p-group and X be a finite-dimensional G-CW complex that is a modulo-p cohomology n-sphere. Then, there is an $m \leq n$ so that X^G is either empty or a modulo-p cohomology m-sphere. If $p \neq 2$, then $n-m$ is even, and even n means X^G is nonempty.*

Proof. Let X be a modulo-p cohomology n-sphere. Then, its cohomology is, with ranks of 1, concentrated in degrees 0 and n, so $\sum_q b_q = 2$. Thus, our dimension counting from the long exact sequences tells us that $\sum_q c_q \leq 2$, i.e., $\sum_q c_q \in \{0, 1, 2\}$. However, as X is a cohomology sphere, we know that $\chi(X) \in \{0, 2\}$, so the formula of Corollary 3.9 means $\sum_q c_q$ can also only be 0 or 2. If we have the former, then X^G is empty, and if we have the latter, then this means X^G has the cohomology of an m-sphere for some $m \leq n$. That m being less than n is clear from the dimension counting inequalities. Finally, observe that when p is odd, the Euler characteristics of n-spheres and m-spheres agree modulo-p exactly when they are equal, so the quantity $n-m$ must be even. Then, if n is even, this forces $\chi(X^G) = 2$ modulo-p, so the fixed point space X^G cannot be empty. This completes the proof. □

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