On noncommutative graphs and Poulin's stabilizer formalism

Dheeran E. Wiggins and Roy Araiza

Department of Mathematics
University of Illinois

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Overview

The contemporary mathematical paradigm for quantum mechanics can be summarized via four axioms.

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Categorical Setting

To clarify our setting, we define a category $\mathsf{Hilb}_{\mathbb{C}}$ with

- *objects*: complex Hilbert spaces $(\mathbb{C} \curvearrowright \mathscr{H}, +, \langle -, - \rangle)$
- *morphisms*: bounded operators $\mathrm{Hom}_{\mathsf{Hilb}_{\mathbb{C}}}(\mathscr{H}, \mathscr{K}) = \mathbb{B}(\mathscr{H} : \mathscr{K})$

We take $\mathsf{Hilb}_{\mathbb{C}}$ to be a "symmetric monoidal, semiadditive †-category." Effectively, this means we can take

- tensor products $\mathscr{H} \otimes \mathscr{K}$
- direct sums $\mathscr{H} \oplus \mathscr{K}$
- adjoints $\mathscr{H} \mapsto \mathscr{H}^{\dagger}$

in the natural ways.

State Space Axiom

Axiom I: State Space

Any quantum system Q is represented by a complex Hilbert space $\mathscr{H}^Q \in \mathsf{Hilb}_{\mathbb{C}}$, called the state space. States of the system are represented by unit-trace, positive semi-definite operators acting on $\mathscr{H}$, called density operators $\mathcal{D}(\mathscr{H}) \subseteq \mathbb{B}(\mathscr{H})$.

Multiple System Axiom

Axiom II: Multiple System

Any pair of quantum systems A and B can be represented as a joint system AB via the tensor product in $\mathsf{Hilb}_{\mathbb{C}}$:

$$\mathcal{H}^{AB} := \mathcal{H}^{A} \otimes \mathcal{H}^{B}.$$

System Evolution Axiom

Axiom III: System Evolution

A quantum system Q undergoing closed evolution is described by a unitary transformation on the state space $\mathcal{H}^Q$.

Remember, a unitary $U \in \mathbb{B}(\mathcal{H}^Q)$ means $UU^\dagger = U^\dagger U = I^Q$.

Measurement Axiom

Axiom IV: Measurement

Every measurement of a finite dimensional quantum system is described by a set of orthogonal projectors $\{P_i\}_{i=1}^r$ such that $\sum_{i=1}^r P_i = I^Q$. If ρ is the state of Q prior to measurement, then with probability $\mathbb{P}(i) = \mathrm{tr}(P_i\rho)$, the post-measurement state is

$$\rho_i = \frac{P_i \rho P_i}{\mathbb{P}(i)}.$$

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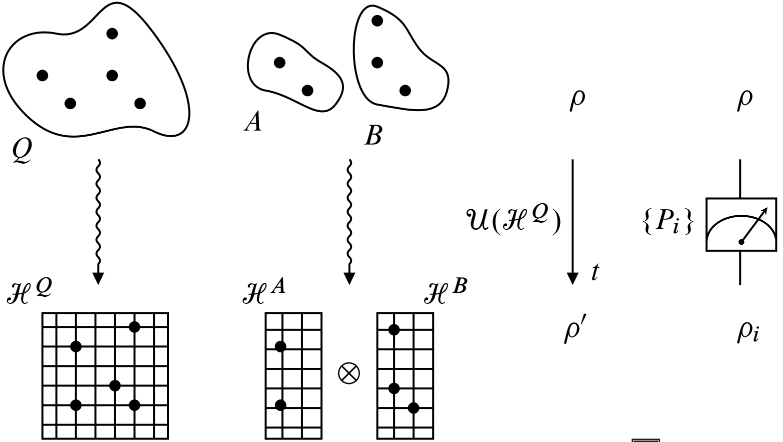
Quantum Axioms Visualized



Pauli Group

We call Hilbert spaces $\mathscr{H} \simeq \mathbb{C}^2$ qubits.

Pauli Group

The *Pauli group* $\mathscr{P}$ is the nonabelian matrix group generated by

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

There is a natural action of $\mathscr{P}$ on a qubit $\mathscr{H}$.

n-Qubit Pauli Group

The *n-qubit Pauli group* $\mathcal{P}_n$ is

$$\mathcal{P}_n := \left\{ i^d \bigotimes_{k=1}^{n} \Sigma_{(k)} : d \in \mathbb{F}_4 \text{ and } \Sigma_{(k)} \in \mathcal{P} \right\} \hookrightarrow \mathrm{GL}_{2^n}(\mathbb{C}).$$

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Denote a 1-local action of $\Sigma \in \mathcal{P}$ on qubit j of $\mathcal{H} \simeq \bigotimes_j \mathbb{C}^2$ by

$$\Sigma_j := I_2 \otimes I_2 \otimes \cdots \underbrace{\otimes \Sigma \otimes}_{j\text{th position}} \cdots \otimes I_2.$$

Then,

$$\mathcal{P}_n = \langle i I_j, X_j, Z_j : 1 \leq j \leq n \rangle$$

If $\mathcal{H} \simeq (\mathbb{C}^2)^{\otimes n}$, then $\Sigma_j \in \mathcal{P}_n$ and $\mathcal{P}_n \curvearrowright \mathcal{H}$.

Error Correction

We model quantum errors as quantum channels.

(i) A *superoperator* is a linear map $\mathcal{E} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$.

(ii) A *quantum channel* $\mathcal{E}$ is a superoperator which is completely positive and trace-preserving.

That is, $\mathcal{E} \otimes \mathrm{id}_k \geq 0$ for all k and $\mathrm{tr}(\mathcal{E}\rho) = \mathrm{tr}(\rho)$.

Theorem (Kraus Representation)

A superoperator $\mathcal{E} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ is completely positive if and only if there are Kraus operators $\{E_i : \mathcal{H} \to \mathcal{K}\}_{i=1}^{r}$ such that

$$\mathcal{E}(-) = \sum_{i=1}^{r} E_i(-)E_i^{\dagger}.$$

In particular, every error has Kraus operators.

Some terminology:

(i) A *codespace* is a subspace $\mathcal{C} \subseteq \mathcal{H}$.

(ii) Given an error $\mathcal{E} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$, we call $\mathcal{R} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ a *recovery channel* if for all states $\rho \in \mathcal{D}(\mathcal{C}) \subseteq \mathbb{B}(\mathcal{C})$,

$$(\mathcal{R} \circ \mathcal{E})(\rho) \propto \rho.$$

(iii) An error $\mathcal{E}$ is *correctable* if a codespace $\mathcal{C}$ and recovery channel $\mathcal{R}$ exist.

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Knill-Laflamme

Theorem (Knill-Laflamme Subspace Condition)

An error $\mathcal{E}$ with Kraus operators $\{E_i\}_{i=1}^r$ is correctable if and only if the projection $P : \mathcal{H} \twoheadrightarrow \mathcal{C}$ onto the codespace admits

$$PE_i^\dagger E_j P = \lambda_{ij} P,$$

for all $1 \leq i, j \leq r$, where $[\lambda_{ij}] \in \mathbb{M}_r(\mathbb{C})$ is self-adjoint.

Gottesman's Stabilizer Formalism

Let $\mathcal{S}$ an abelian subgroup $\langle S_1, \ldots, S_s \rangle \leq \mathcal{P}_n$ without $-I^{\otimes n}$. Then, $\mathcal{S}$ is a stabilizer. We can form a *stabilizer codespace*

$$\mathcal{C} \equiv \mathcal{C}(\mathcal{S}) := \operatorname{span}_{\mathbb{C}} \left\{ v \in (\mathbb{C}^2)^{\otimes n} : S_j v = v \text{ for all } 1 \leq j \leq s \right\}.$$

Theorem (Stabilizer Formalism)

An error $\mathcal{E}$ with Kraus operators $\{E_i\}_{i=1}^r$ is correctable on $\mathcal{C}(\mathcal{S})$ if and only if for all $1 \leq i, j \leq r$,

$$E_i^\dagger E_j \in \operatorname{span}_{\mathbb{C}} \{ (\mathcal{P}_n \setminus \mathcal{N}_{\mathcal{P}_n}(\mathcal{S})) \cup \mathcal{S} \}.$$

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Operator Quantum Error Correction

Suppose we have a decomposition

$$\mathcal{H} \simeq \underbrace{(\mathcal{H}^A \otimes \mathcal{H}^B)}_{\mathcal{C}} \oplus \mathcal{C}^\perp .$$

Let $\mathcal{E}$ be an error. We call $\mathcal{H}^A$ noiseless if for all $\rho^A \in \mathbb{B}(\mathcal{H}^A)$ and $\rho^B \in \mathbb{B}(\mathcal{H}^B)$,

$$\mathcal{E}(\rho^A \otimes \rho^B) = \rho^A \otimes \tau^B$$

for some $\tau^B \in \mathbb{B}(\mathcal{H}^B)$. Correctability is defined on the A system.

Poulin's Stabilizer Formalism

Form a quotient of $\mathbb{B}(\mathcal{C})$ to define the gauge group $\mathcal{G}$ of operators:

$$\rho \sim \rho' \quad \Longleftrightarrow \quad (\exists g \in \mathcal{G})\big(\rho = g\rho' g^{\dagger}\big)$$

Theorem (Stabilizer Formalism)

Given an error $\mathcal{E}$ on $\mathcal{H} \simeq (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{C}^{\perp}$ with Kraus operators $\{E_i\}_{i=1}^{r}$, a recovery $\mathcal{R}$ exists if and only if for all $1 \leq i, j \leq r$,

$$E_i^{\dagger} E_j \in \operatorname{span}_{\mathbb{C}} \{(\mathcal{P}_n \setminus \mathcal{N}_{\mathcal{P}_n}(\mathcal{S})) \cup \mathcal{G}\}.$$

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Knill-Laflamme, Reformulated

Winter Space/Noncommutative Graph

Let $\mathcal{E}$ be an error channel with Kraus operators $\{E_i\}_{i\in I}$. Then, the *Winter space* (or *noncommutative graph*) of the channel is the space

$$\mathcal{V}_{\mathcal{E}} := \operatorname{span}_{\mathbb{C}} \left\{ E_i^\dagger E_j : i, j \in I \right\}.$$

We can rephrase Knill-Laflamme as

$$P \mathcal{V}_{\mathcal{E}} P = \mathbb{C} P,$$

meaning $\mathcal{C}$ is a codespace if and only if $\dim P \mathcal{V}_{\mathcal{E}} P = 1$.

Operator Systems

An operator system (os) is a subspace $\mathcal{V} \subseteq \mathbb{B}(\mathcal{H})$ so that $I \in \mathcal{V}$ and $v \in \mathcal{V}$ implies $v^\dagger \in \mathcal{V}$.

Theorem (Duan 09)

A subspace $\mathcal{V} \subseteq \mathbb{B}(\mathcal{H})$ is a noncommutative graph $\mathcal{V}_\mathcal{E}$ for some channel $\mathcal{E}$ if and only if it is an os.

Recovering Gottesman's Formalism

Theorem (Araiza et al. 24)

Let $G \subseteq \mathcal{P}_n$ be an abelian subgroup so that $-I^{\otimes n} \notin G$ and $M_{\mathbf{0}} \in \mathbb{M}_{2^n}(\mathbb{C})$. Let

$$\mathcal{V}_{M_0} := \operatorname{span}\{gM_{\mathbf{0}}g : g \in G\}$$

be the noncommutative graph. Then,

$$\operatorname{span}\{\mathcal{V}_{M_0} : M_{\mathbf{0}} \text{ makes } \mathcal{V}_{M_0} \text{ os}\} = \operatorname{span}\left\{(\mathcal{P}_n \setminus \mathcal{N}_{\mathcal{P}_n}(G)) \cup I^{\otimes n}\right\}.$$

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Recovering Poulin's Formalism

Let $\mathcal{G} \subseteq \mathcal{P}_n$ be the gauge subgroup, in the sense of Poulin, associated to a noise channel $\mathcal{E}$ and $\boldsymbol{M_0} \in \mathbb{M}_{2^n}(\mathbb{C})$. Then,

$$\operatorname{span}\{\mathcal{V}_{\boldsymbol{M}_0} : \boldsymbol{M_0} \text{ makes } \mathcal{V}_{\boldsymbol{M}_0} \text{ os}\} = \operatorname{span}\left\{(\mathcal{P}_n \setminus \mathcal{Z}_{\mathcal{P}_n}(\mathcal{G})) \cup I^{\otimes n}\right\}.$$

Sketch of Proof

Poulin deduces an explicit set of generators

$$\mathcal{G} \simeq \langle i, Z_1, \ldots, Z_s, X_{s+1}, Z_{s+1}, \ldots, X_{s+r}, Z_{s+r} \rangle .$$

- Write M_0 in the Pauli basis.
- Form an indicator function Ξ which outputs 1 if the $\mathcal{G}$-elements commute with the basis elements in M_0's Pauli expansion, and -1 otherwise.
- Separate the sum into the $\mathcal{Z}_{\mathcal{P}_n}(\mathcal{G})$ and $\mathcal{P}_n \setminus \mathcal{Z}_{\mathcal{P}_n}(\mathcal{G})$ cases.
- Pick coefficients to get $\mathcal{V}_{M_0}$ to be unital.
- Span over $\mathbb{C}$ to get the result.

Heisenberg-Weyl Group

We may wish to generalize $\mathcal{P}_n$ to act on n-qudits $(\mathbb{C}^d)^{\otimes n}$.
Define $\mathcal{P}_{d,n}$ to be $\langle \sqrt{\omega} I_j, X_j, Z_j : 1 \leq j \leq n \rangle$, where

$$\text{“shift”} \quad X : \sum_{k \in \mathbb{Z}/d} e_k e_k^\dagger \mapsto \sum_{k \in \mathbb{Z}/d} e_{k+1} e_k^\dagger,$$

$$\text{“clock”} \quad Z : \sum_{k \in \mathbb{Z}/d} e_k e_k^\dagger \mapsto \sum_{k \in \mathbb{Z}/d} \omega^k e_k e_k^\dagger,$$

ω is the dth root of unity, and e_k is the kth standard basis vector.

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Full Generality

Replacing $\mathcal{P}_n$ with $\mathcal{P}_{d,n}$, taking the analogue of $\mathcal{G}$, and finding $M_0 \in \mathbb{M}_{d^n}(\mathbb{C})$, the same characterization of Poulin's stabilizer formalism via Winter spaces holds.

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