PROBLEM SET 04: STATES AND QUANTUM CIRCUITS

Exercise 0.1 (Recalling Defintions)**.** Define a

(i) unitary operator U on a Hilbert space $\mathscr{H}$.
(ii) pure state.
(iii) quantum circuit U.
(iv) function $f : G \to S$ which separates H-cosets.

Exercise 0.2. Let A and B be single-qubit quantum systems. Let AB denote their joint system, which thus has state space $\mathscr{H}^{AB} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$. Define the 00th Bell basis vector $|\Phi_{00}\rangle \in \mathscr{H}^{AB}$ by

$$|\Phi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

(i) Prove that $\Phi = |\Phi_{00}\rangle\langle\Phi_{00}|$ is a (pure) state. That is, show that
  (a) $|\Phi_{00}\rangle$ is a unit vector.
  (b) $\operatorname{tr} \Phi = 1$.
  (c) for all $|\psi\rangle \in \mathscr{H}^{AB}$, we have $\langle\psi|\Phi|\psi\rangle \geq 0$.
(ii) Show that $|\Phi_{00}\rangle\langle\Phi_{00}|$ cannot be written in the form $|\varphi_1\rangle \otimes |\varphi_2\rangle$, where $|\varphi_1\rangle \in \mathscr{H}^A \simeq \mathbb{C}^2$ and $|\varphi_2\rangle \in \mathscr{H}^B \simeq \mathbb{C}^2$. This phenomenon is what we call *entanglement*.

Exercise 0.3. Let Q be an arbitrary quantum system; let $\mathscr{H}^Q$ denote its state space. Prove that closed evolution of a density operator $\rho \in \mathscr{D}(\mathscr{H})^Q$ is reversible.

Exercise 0.4. Read the *abstract* of Kuperberg's paper on the dihedral hidden subgroup problem.[1] Write up a brief summary of the content of this section.

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[1]See the course page for a link.