PROBLEM SET 02: LINEAR SPACES

Exercise 0.1 (Recalling Defintions)**.** Define a

(i) vector space $\mathcal{V}$.
(ii) ket $|\psi\rangle$ and a bra $\langle\psi|$
(iii) inner product $\langle - | - \rangle$.
(iv) dual space $\mathcal{V}^*$.

Exercise 0.2 (Finite Spaces are $\mathbb{C}^n$)**.** Let $\mathcal{V}$ be a $\mathbb{C}$-linear space with basis $\beta = \{v_1, \dots, v_n\}$. Then, $\dim \mathcal{V} = |\beta| = n$. Show that the vector space homomorphism $\varphi : \mathcal{V} \to \mathbb{C}^n$ given by

$$\varphi(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

is an isomorphism $\mathcal{V} \simeq \mathbb{C}^n$. That is, check that φ is bijective.

Exercise 0.3 (Completeness)**.** Let $(\mathcal{H}, (-,-))$ be an inner product space. Now, for some definitions.

(i) Let
$$d : \mathcal{H} \times \mathcal{H} \to [0, \infty)$$
be the *distance function* defined by
$$d(v, w) = \sqrt{(v - w, v - w)}.$$
(ii) We say that a sequence $\{h_1, h_2, \dots\}$ of vectors in $\mathcal{H}$ is *Cauchy* if for all $\varepsilon > 0$, there is a positive integer $N \in \mathbb{Z}_+$ such that for all positive integers $n, m > N$,
$$d(h_n, h_m) = \sqrt{(h_n - h_m, h_n - h_m)} < \varepsilon.$$
(iii) We say that $(\mathcal{H}, (-,-))$ is *complete* with respect to d if the following statement[1] holds for every Cauchy sequence $\{h_1, h_2, \dots\}$ in $\mathcal{H}$: there exists a vector $h \in \mathcal{H}$ so that for all $\varepsilon > 0$, there exists a positive integer $N \in \mathbb{Z}_+$ such that for all positive integers $n > N$,
$$d(h_n, h) = \sqrt{(h_n - h, h_n - h)} < \varepsilon.$$

Prove that if $\dim \mathcal{H} = n < \infty$, so that $(\mathcal{H}, (-,-))$ is a finite dimensional inner product space, then $\mathcal{H}$ is always complete with respect to d.

Definition 0.4 (Hilbert Space)**.** A Hilbert space is an inner product space which is complete with respect to the distance function induced by its inner product.

Thus, by Exercise 0.3, every finite dimensional inner product space is a Hilbert space. Furthermore, by Exercise 0.2, every such space is isomorphic to $\mathbb{C}^n$, where n is the space's dimension. In particular, this justifies why our study of finite dimensional quantum systems amounts to a study of $\mathbb{C}^n$ with the usual inner product structure.

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[1]That is, $\lim_{n\to\infty} h_n = h$ with respect to the distance function d.