PROBLEM SET 01: GROUP BASICS

Exercise 0.1 (Recalling Defintions)**.** Define a

(i) group $(G, \cdot)$.
(ii) finitely generated abelian group $(G, \cdot)$.
(iii) normal subgroup $H \lhd (G, \cdot)$.
(iv) group homomorphism $\varphi : (G_1, \cdot) \to (G_2, *)$.

Exercise 0.2 (Subgroups)**.** Prove that

(i) the set $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$.
(ii) every subgroup H of an abelian group $(G, \cdot)$ is normal.

Conclude that we can define a quotient group $\mathbb{Z}/n\mathbb{Z}$.

Exercise 0.3 (Direct Products)**.** Let $(G_1, \cdot)$ and $(G_2, *)$ be groups. The (direct) product $G_1 \times G_2$ was defined with an underlying set given by the Cartesian product

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1 \text{ and } g_2 \in G_2\}$$

and with an operation $(-) \star (-) : (G_1 \times G_2)^2 \to G_1 \times G_2$ given by

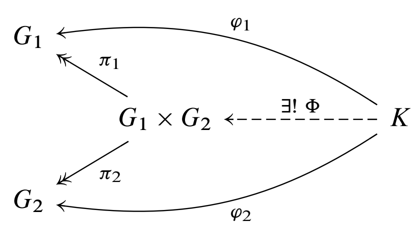
$$(g_1, g_2) \star (g_1', g_2') = (g_1 \cdot g_1', g_2 * g_2').$$

Show that $(G_1 \times G_2, \star)$ does, in fact, define a group (i.e., check the three axioms).

Now, let e_1 and e_2 be the identity elements of G_1 and G_2, respectively. Observe that there is a surjective homomorphism $\pi_1 : G_1 \times G_2 \to G_1$ which sends an element (g_1, g_2) to (g_1, e_2). Likewise for $\pi_2 : G_1 \times G_2 \to G_2$. We call these π_j the *natural projections*, writing $\pi_j : G_1 \times G_2 \twoheadrightarrow G_j$. Let $(K, +)$ be an arbitrary group and $\varphi_1 : K \to G_1$ and and $\varphi_2 : K \to G_2$ be homomorphisms. Show that there is a homomorphism $\Phi : K \to G_1 \times G_2$ such that for all $k \in K$,

$$\pi_j(\Phi(k)) = \varphi_j(k), \quad j \in \{1, 2\}.$$

It is then a simple check to show that Φ is actually the *unique* such homomorphism. This property is called the *universal property of products* for groups, which we may summarize using the following (commutative) diagram:



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