Topological (Quantum) Field Theories
in Quantum Computing

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Math⊗Quantum

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Overview

Quantum computation often asks us to answer the following.

What is a qubit?

In the lab, there are many possible choices.

Spin systems, photons, superconducting circuits, trapped ions, . . .

But as theorists, we have a pretty clear answer.

A qubit consists of vectors of the form

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

If we consider all superpositions, we get a “space” of vectors we call $\mathbb{C}^2$.

This is the Hilbert space associated to a qubit q_0.

Let us draw a silly picture of this model.

$$q_0 : \mathbb{C}^2 \ \text{———————}$$

What about two qubits q_0 and q_1?

The associated Hilbert space is the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$.

$$q_0 : \mathbb{C}^2 \;\; \text{———————}$$
$$q_1 : \mathbb{C}^2 \;\; \text{———————}$$

More generally, say we have a quantum system consisting of n qubits.

Well, our picture now looks like

$$\begin{array}{r} q_0 : \mathbb{C}^2 \;\text{———————} \\ q_1 : \mathbb{C}^2 \;\text{———————} \\ \vdots \qquad\qquad\qquad\qquad \\ q_{n-1} : \mathbb{C}^2 \;\text{———————} \end{array}$$

So, the Hilbert space associated to n qubits should be

$$\mathbb{C}^2 \underbrace{\otimes \cdots \otimes}_{n} \mathbb{C}^2.$$

As we will see, this game of assigning Hilbert spaces to quantum systems based on tensor products is exactly what a topological quantum field theory is.

In the most generality, a Hilbert space is defined as follows.

Definition

A Hilbert space $\mathscr{H}$ is a complex inner product space that is complete with respect to the induced metric.

In finite dimensions, our Hilbert space is always $\mathscr{H} \simeq \mathbb{C}^n$. All this means is we care about vectors

$$
\begin{aligned}
|\psi\rangle &= \alpha_1 |0\rangle + \alpha_2 |1\rangle + \cdots + \alpha_n |n-1\rangle \\
|\varphi\rangle &= \beta_1 |0\rangle + \beta_2 |2\rangle + \cdots + \beta_n |n-1\rangle
\end{aligned}
$$

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. . . and being able to take inner products

$$\langle \psi | \varphi \rangle = \sum_{i=1}^{n} \alpha_i^* \beta_i \,.$$

In infinite dimensions, we care about the same sort of vectors, except now there are infinitely many terms! This means we have to be careful about when things converge.

An example of an infinite dimensional Hilbert space that crops up all the time in physics is L^2 space.

This is our space of all wave functions $\Psi(x,t)$.

Challenge: How could we give L^2 space an inner product?

For quantum computation (and for topological field theories), we "often" only care about the finite dimensional case.

Hilbert spaces also facilitate something like "time-reversal."

Morally, if we apply some quantum circuit to our state, we may also like to undo it!

Given an operator

$$M : \mathcal{H} \to \mathcal{K}$$

there is a dagger (or adjoint) operator

$$M^{\dagger} : \mathcal{K} \to \mathcal{H}.$$

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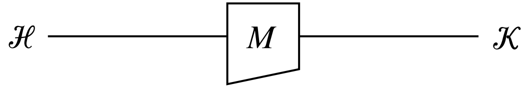
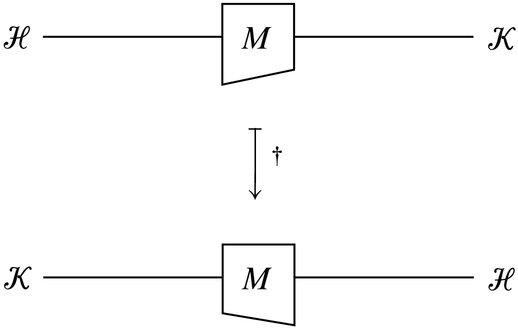
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

If we draw an operator as



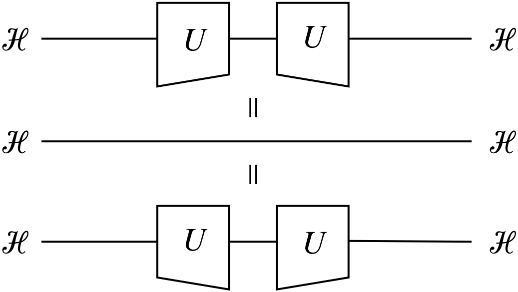
Then daggering looks like



A quantum circuit is an operator $U : \mathscr{H} \to \mathscr{H}$ such that

$$U^\dagger U = \mathbb{1} = U U^\dagger$$

Pictorially, this means



"Performing a circuit and then undoing it does nothing."

To reason about quantum information processes mathematically, we tend to restrict ourselves using some axioms.

Let us consider an arbitrary quantum system Q.

The state space axiom says we model the quantum system Q by assigning to it a Hilbert space $\mathscr{H}_Q$.

For example, when Q is a collection of n qubits, we saw that

$$\mathcal{H}_Q \simeq \mathbb{C}^2 \underbrace{\otimes \cdots \otimes}_{n} \mathbb{C}^2.$$

Now, say our system $Q(t)$ is time-dependent.

How would we represent time evolution

$$Q(t_0) \longrightarrow Q(t_1)?$$

Well by the state space axiom, we have a Hilbert space $\mathcal{H}_{Q(t_0)}$ and a Hilbert space $\mathcal{H}_{Q(t_1)}$.

So, the evolution axiom says we model evolution using operators

$$\mathcal{H}_{Q(t_0)} \xrightarrow{\quad M \quad} \mathcal{H}_{Q(t_1)}$$

Moreover, if the evolution is closed/reversible, we ask that our operator be a quantum circuit. This property is called unitarity.

Suppose now that we want to combine or correlate two systems.

How should we go about modeling this?

Let A be Alice's system and B be Bob's. To consider them as a single quantum system, we want to consider the joint system

$$A \amalg B.$$

The multiple systems axiom says that the state space associated to the joint system of Alice and Bob is just the tensor product of their state spaces:

$$\mathcal{H}_{A \amalg B} \simeq \mathcal{H}_A \otimes \mathcal{H}_B .$$

$$\mathcal{H}_{A \amalg B} \;\text{———————}\; = \; \begin{matrix} \mathcal{H}_A \;\text{———————} \\ \mathcal{H}_B \;\text{———————} \end{matrix}$$

This modeling process given by the quantum postulates reflects a change in setting

$$\mathscr{Q}\text{uant} \to \mathscr{H}\text{ilb}.$$

What is a setting?

Well, we should have

(i) things "objects."
(ii) processes "morphisms."
(iii) successive processes "composition."

Formally, we usually call settings categories.

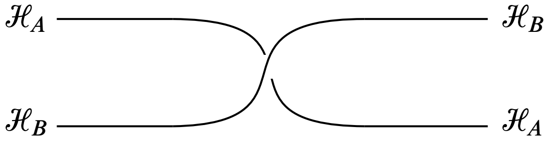
Then, we call a change of setting a functor.

Sometimes we can equip our setting with a way to join objects and processes:

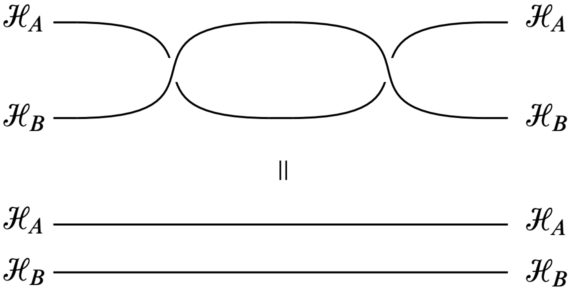
(i) in $\mathscr{H}\mathrm{ilb}$ we have $\otimes$.

(ii) in $\mathscr{Q}\mathrm{uant}$ we have $\amalg$.

In fact, we can sometimes swap joined objects:



In our two settings, swapping twice does nothing ("symmetric"):



If both our settings have a joining operation that allows symmetric swaps, we say the change of setting is a symmetric monoidal functor.

A functorial quantum field theory is a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{Q}\mathrm{uant} \to \mathcal{H}\mathrm{ilb}.$$

That is, $\mathcal{Z}$ is a change in setting that respects the joining operation.

We can choose all sorts of models for $\mathcal{Q}$uant.

In typical topological field theory (in dimension $d+1$), we let

$$\mathcal{Q}\mathrm{uant} = \mathcal{B}\mathrm{ord}_{d+1}.$$

The monoidal category $\mathcal{B}\mathrm{ord}_{d+1}$ has

(i) objects that are d-dimensional physical spaces.

(ii) processes given by ($d + 1$-dimensional) "spacetimes."

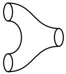
(iii) composition is gluing.

(iv) joining is just using $\amalg$.

Let us consider some examples of these processes in dimension 2.

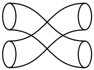
Here are some classic spacetimes between circles:

- pair of pants (2 to 1)



- cup (0 to 1)

- swap (2 to 2)



So, a topological quantum field theory in dimension $d + 1$ is a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{B}\mathrm{ord}_{d+1} \to \mathcal{H}\mathrm{ilb}.$$

(Atiyah, Segal)

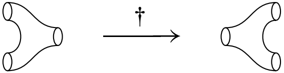
Notice that we have not said anything about closed evolution.

To encode unitarity into a topological field theory, we can add a dagger to give us time reversal (Selinger).

For $\mathcal{H}$ilb, we already saw the dagger!

For $\mathscr{B}\mathrm{ord}_{d+1}$, we could say the dagger is literal reversal.

In dimension 2, this looks like



Then a unitary topological field theory is a symmetric monoidal †-functor

$$\mathcal{Z} : \mathcal{B}\mathrm{ord}_{d+1} \to \mathcal{H}\mathrm{ilb}$$

This is a change in setting that respects the joining operation and the † operation.

Since $\mathcal{Z}$ respects the $\dagger$, it takes reversible evolutions to quantum circuits. As you may expect, these are often the topological field theories we care about in quantum information science.

So. . . , where can we go from here?

Well, we saw that the pictures we drew of our modeling processes gave us the notation for quantum circuits.

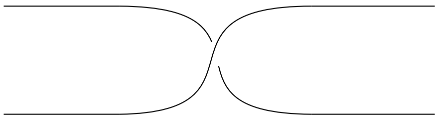
We can study all sorts of settings with various graphical calculi.

Quantum computing researchers use tensor networks. These families of graphical calculi precisely stem from the string diagrams we have been drawing for $\mathcal{H}$ilb (see Heunen-Vicary for more on this).

These diagrams give an interesting unifying perspective on quantum theory.

For example, Feynman diagrams in particle physics are exactly the graphical calculus of a certain monoidal category (see Baez-Lauda for more on this).

Recall that we often have a way to swap systems:



Sometimes this braiding operation is not symmetric, so doing it twice does not return us to our original state.

It turns out that in the setting of anyons, we can use these braidings and "fusion rules" to encode our quantum gates (see Burton, Preskill, Kawagoe for more on this).

Another direction is to consider alternative models for $\mathcal{Q}\mathrm{uant}$.

I am particularly interested in considering the setting of defect bordisms

$$\mathcal{Q}\mathrm{uant} = \mathcal{D}\mathrm{ef}_2.$$

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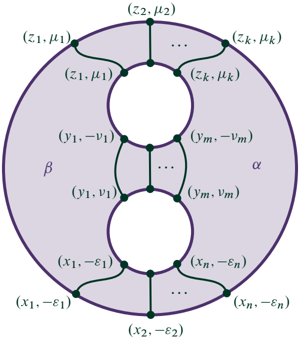
Here is one defect pair of pants in $\mathscr{D}\mathrm{ef}_2$:



It turns out,[1] the resultant (dimension 2) defect field theories

$$\mathcal{Z} : \mathcal{D}\mathrm{ef}_2 \to \mathcal{H}\mathrm{ilb}$$

give rise to gadgets called pivotal 2-categories. These are settings with "processes between processes."

[1] This construction is due to Davydov, Kong, and Runkel (link). I learned it from some notes by Nils Carqueville (link).

Currently, I am thinking about various properties of these pivotal 2-categories when we, say, make the theory unitary (have $\mathcal{Z}$ preserve daggers).

These ideas naturally lead to some classification problems.

Further reading:

(BL) John Baez and Aaron Lauda's chapter *A Prehistory of n-Categorical Physics* (link).

(B1) John Baez's *Quantum Quandaries: a Category-Theoretic Perspective* (link).

(B2) Simon Burton's note *A Short Guide to Anyons and Modular Functors* (link).

(K) Kyle Kawagoe's lecture notes *Fusion Categories and Condensed Matter* (link).

(P) John Preskill's chapter *Topological Quantum Computation* (link).

(HV) Chris Heunen and Jamie Vicary's notes *Introduction to Categorical Quantum Mechanics* (link).

(CR) Nils Carqueville and Ingo Runkel's notes *Introductory lectures on topological quantum field theory* (link).

(K) Joachim Kock's book *Frobenius algebras and 2D topological quantum field theories* (link).

(F) Dan Freed's course notes *Bordism: Old and New* (link).

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Every sufficiently good analogy is yearning to become a functor.

– John Baez

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