Some (semi)simple things about 2D unitary TQFT with defects

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Overview

When thinking about quantum information, we often restrict ourselves via a few axioms/postulates.

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(A1) State space: To each quantum system Q, associate a (finite-dimensional) Hilbert space $\mathcal{H}_Q$.

Categorically, (A1) tells us we want functors

$$\mathcal{Z} : \mathfrak{Quant} \to \mathfrak{FdHilb}.$$

(A1) State space: To each quantum system Q, associate a (finite-dimensional) Hilbert space $\mathcal{H}_Q$.

(A2) Multiple systems: To each joint system $AB = A \amalg B$, associate the tensor product of their Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$.

Assuming "joining systems" is a monoidal product $\amalg$ on $\mathfrak{Q}$uant, (A1)$\wedge$(A2) asks for monoidal functors

$$\mathcal{Z} : (\mathfrak{Q}\text{uant}, \amalg) \to (\mathfrak{F}\text{dHilb}, \otimes).$$

Really, we should also be able to swap A and B coherently, so that swapping twice gets us back to AB.

So, we want *symmetric* monoidal functors

$$\mathcal{Z} : (\mathfrak{Quant}, \amalg, \mathrm{swap}) \to (\mathfrak{F}\mathrm{dHilb}, \otimes, \mathbb{F}_{\boldsymbol{AB}}).$$

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(A1) State space: To each quantum system Q, associate a (finite-dimensional) Hilbert space $\mathcal{H}_Q$.

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(A3) System evolution: Closed evolution $A \to B$ of quantum systems corresponds to unitary operators $U : \mathcal{H}_A \to \mathcal{H}_B$.

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Since (A3) corresponds to reversibility, we need a reversal action on both $\mathfrak{Quant}$ and $\mathfrak{FdHilb}$.

Let $\dagger : \mathfrak{Quant} \to \mathfrak{Quant}$ be the time-reversal action.

This operation

(i) reverses trajectories.

(ii) does nothing to quantum systems.

(iii) is its own inverse.

That is, $\dagger$ is an involutive contravariant endofunctor that is the identity on objects.

For Hilbert spaces, we already know what † should be: taking adjoints.

Combining all three, (A1)∧(A2)∧(A3) suggests we look at dagger symmetric monoidal functors

$$\mathcal{Z} : (\mathfrak{Q}\text{uant}, \amalg, \text{swap}, \dagger) \to (\mathfrak{F}\text{dHilb}, \otimes, \mathbb{F}_{\boldsymbol{AB}}, \dagger).$$

What is Quant?

Atiyah and Segal (1988, 2004, and more): a category of bordisms.

Generally, we can define

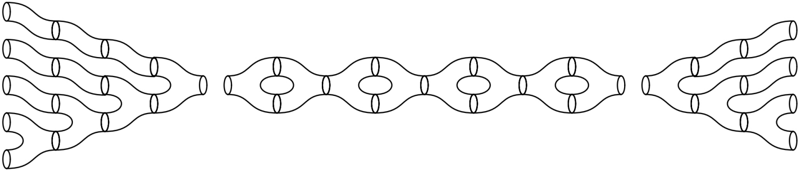
$$\mathfrak{B}\mathrm{ord}_{d+1} = \begin{cases} \text{objects: closed oriented } d\text{-manifolds} \\ \text{morphisms: diffeomorphism (rel } \partial\text{) classes of bordisms} \end{cases}$$

Composition is gluing.

For example, in two-dimensions,

$$\mathfrak{B}\mathrm{ord}_2 = \begin{cases} \text{objects: } \mathbb{S}^1 \amalg \cdots \amalg \mathbb{S}^1 \\ \text{morphisms: compact surfaces between them} \end{cases}$$

Here is a bordism $(\mathbb{S}^1)^{\sqcup 5} \to (\mathbb{S}^1)^{\sqcup 4}$ (Kock):



Then, an (A1)$\wedge$(A2) functor is called a two-dimensional topological field theory.

If we add (A3), we get a *unitary* theory.

What are two-dimensional topological field theories?

Folklore (Dijkgraaf, others): they are commutative Frobenius algebras.

See Sawin or Kock for proof.

Frobenius algebra:

(i) $\mathbb{C}$-vector space $\mathfrak{A}$

(ii) has multiplication $\mathcal{Z}($$)$ and comultiplication $\mathcal{Z}($$)$

(iii) satisfies Frobenius relation

What about unitary two-dimensional topological field theories?

Durhuus and Jonsson (1994): they are classified by $\dim(\mathcal{Z}(\mathbb{S}^1))$ many positive real numbers.

Really, this means unitary two-dimensional theories are *semisimple* commutative Frobenius algebras.

How to prove this?

(i) The Frobenius algebra $\mathcal{Z}(\mathbb{S}^1)$ is a *dagger* Frobenius algebra: $\mathcal{Z}($$)^\dagger = \mathcal{Z}($$)$ and likewise for (co)units.

(ii) Commutative dagger Frobenius algebras are precisely commutative H^*-algebras (Heunen-Vicary).

(iii) H^*-algebras are semisimple (Ambrose).

(iv) Commutativity puts constraint on dimension.

Let's add defects (or domain walls, seams) to our field theories.

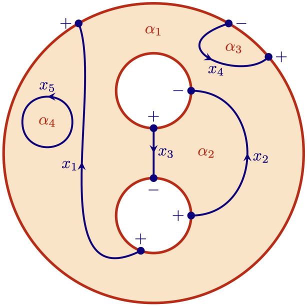
We will need a defect version of $\mathfrak{Bord}_2$.

(i) Take some defect data $\mathbb{D}$.

(ii) Decorate edges of circles with points and bordisms with line defects.

(iii) Call the resulting bordism category $\mathfrak{B}\mathrm{ord}_2^{\mathrm{def}}(\mathbb{D})$.

Here is a defect bordism (Carqueville):



A two-dimensional topological field theory with defects is a symmetric monoidal functor

$$\mathcal{Z} : \mathfrak{Bord}_2^{\mathrm{def}}(\mathbb{D}) \to \mathfrak{F}\mathrm{dHilb},$$

with the usual symmetric monoidal structure.

Given a theory, Davydov, Kong, and Runkel build a pivotal 2-category $\mathfrak{B}_{\mathcal{Z}}$:

(i) objects: 2-dimensional regions on bordisms
(ii) 1-morphisms: 1-dimensional line defects
(iii) 2-morphisms: junction points/local operators
(iv) adjoints for 1-morphisms: orientation reversal

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This is not a classification (yet), since we need to go in the opposite direction.

These defect theories satisfy (A1) and (A2).

So, what if we make them unitary (A3)?

Surely, they should be "semisimple."

Douglas and Reutter (2018): a semisimple 2-category

(i) has adjoints for 1-morphisms.
(ii) is locally semisimple.
(iii) is additive.
(iv) is idempotent complete.

Idea: $\mathfrak{B}_{\mathcal{Z}}$ should be (at least, almost) semisimple:

(i) We already have adjoints for 1-morphisms.

(ii) Baez (1996): By giving a $\mathfrak{F}$dHilb-category a nice antilinear map $\dagger : \mathfrak{B}_{\mathcal{Z}}(\alpha, \beta) \to \mathfrak{B}_{\mathcal{Z}}(\beta, \alpha)$, we can obtain semisimplicity ("2-Hilbert spaces").

(iii) ...

(iv) We should have local idempotent completeness by (ii).

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Further reading:

(C) Nils Carqueville's *Lecture notes on 2-dimensional defect TQFT*

(K) Joachim Kock's book *Frobenius Algebras and 2D Topological Quantum field Theories*

(HV) Chris Heunen and Jamie Vicary's book *Categories for Quantum Theory: An Introduction*

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Thanks! Questions?

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