

SOME (SEMI)SIMPLE THINGS ABOUT 2D UNITARY TQFT WITH DEFECTS

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MAY 01, 2026



PEOPLE

- James Pascaleff
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OVERVIEW

- 1 Axioms to Functors
- 2 Frobenius Algebras and Semisimplicity
- 3 Defects and the 2-Category
- 4 Semisimplicity, Again
- 5 Final Remarks



When thinking about quantum information, we often restrict ourselves via a few axioms/postulates.

(AI) State space: To each quantum system Q , associate a (finite-dimensional) Hilbert space \mathcal{H}_Q .

Categorically, (A_1) tells us we want functors

$$\mathcal{Z} : \text{Quant} \rightarrow \mathcal{F}d\text{Hilb}.$$

- (A₁) State space: To each quantum system Q , associate a (finite-dimensional) Hilbert space \mathcal{H}_Q .
- (A₂) Multiple systems: To each joint system $AB = A \amalg B$, associate the tensor product of their Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$.

Assuming “joining systems” is a monoidal product Π on \mathbf{Quant} ,
 $(A_1) \wedge (A_2)$ asks for monoidal functors

$$\mathcal{Z} : (\mathbf{Quant}, \Pi) \rightarrow (\mathcal{FdHilb}, \otimes).$$

Really, we should also be able to swap A and B coherently, so that swapping twice gets us back to AB .

So, we want *symmetric* monoidal functors

$$\mathcal{Z} : (\text{Quant}, \Pi, \text{swap}) \rightarrow (\mathcal{F}d\text{Hilb}, \otimes, \mathbb{F}_{AB}).$$

- (A₁) State space: To each quantum system Q , associate a (finite-dimensional) Hilbert space \mathcal{H}_Q .
- (A₂) Multiple systems: To each joint system $AB = A \amalg B$, associate the tensor product of their Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$.
- (A₃) System evolution: Closed evolution $A \rightarrow B$ of quantum systems corresponds to unitary operators $U : \mathcal{H}_A \rightarrow \mathcal{H}_B$.

Since (A_3) corresponds to reversibility, we need a reversal action on both Quant and $\mathcal{F}d\text{Hilb}$.

Let $\dagger : \mathbf{Quant} \rightarrow \mathbf{Quant}$ be the time-reversal action.

This operation

- (i) reverses trajectories.
- (ii) does nothing to quantum systems.
- (iii) is its own inverse.

That is, \dagger is an involutive contravariant endofunctor that is the identity on objects.

For Hilbert spaces, we already know what \dagger should be: taking adjoints.

Combining all three, $(A_1) \wedge (A_2) \wedge (A_3)$ suggests we look at dagger symmetric monoidal functors

$$\mathcal{Z} : (\text{Quant}, \Pi, \text{swap}, \dagger) \rightarrow (\mathcal{F}\text{dHilb}, \otimes, \mathbb{F}_{AB}, \dagger).$$

What is $\mathcal{Q}uant$?

Atiyah and Segal (1988, 2004, and more): a category of bordisms.

Generally, we can define

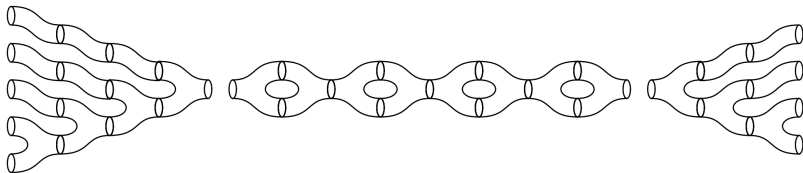
$$\mathcal{B}ord_{d+1} = \left\{ \begin{array}{l} \text{objects: closed oriented } d\text{-manifolds} \\ \text{morphisms: diffeomorphism (rel } \partial) \text{ classes of bordisms} \end{array} \right.$$

Composition is gluing.

For example, in two-dimensions,

$$\mathcal{B}ord_2 = \begin{cases} \text{objects: } \mathbb{S}^1 \amalg \dots \amalg \mathbb{S}^1 \\ \text{morphisms: compact surfaces between them} \end{cases}$$

Here is a bordism $(\mathbb{S}^1)^{\amalg 5} \rightarrow (\mathbb{S}^1)^{\amalg 4}$ (Kock):



Then, an $(A_1) \wedge (A_2)$ functor is called a two-dimensional topological field theory.

If we add (A_3) , we get a *unitary* theory.

What are two-dimensional topological field theories?

Folklore (Dijkgraaf, others): they are commutative Frobenius algebras.

See Sawin or Kock for proof.

Frobenius algebra:

- (i) \mathbb{C} -vector space \mathcal{A}
- (ii) has multiplication $\mathcal{Z}(\triangleright)$ and comultiplication $\mathcal{Z}(\triangleleft)$
- (iii) satisfies Frobenius relation

What about **unitary** two-dimensional topological field theories?

Durhuus and Jonsson (1994): they are classified by $\dim(\mathcal{Z}(\mathcal{S}^1))$ many positive real numbers.

Really, this means unitary two-dimensional theories are *semisimple* commutative Frobenius algebras.

How to prove this?

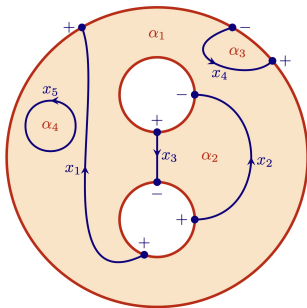
- (i) The Frobenius algebra $\mathcal{Z}(\mathbb{S}^1)$ is a *dagger* Frobenius algebra:
 $\mathcal{Z}(\text{cup})^\dagger = \mathcal{Z}(\text{cap})$ and likewise for (co)units.
- (ii) Commutative dagger Frobenius algebras are precisely commutative H^* -algebras (Heunen-Vicary).
- (iii) H^* -algebras are semisimple (Ambrose).
- (iv) Commutativity puts constraint on dimension.

Let's add **defects** (or domain walls, seams) to our field theories.

We will need a defect version of $\mathcal{B}ord_2$.

- (i) Take some defect data \mathbb{D} .
- (ii) Decorate edges of circles with points and bordisms with line defects.
- (iii) Call the resulting bordism category $\mathcal{Bord}_2^{\text{def}}(\mathbb{D})$.

Here is a defect bordism (Carqueville):



A two-dimensional topological field theory with defects is a symmetric monoidal functor

$$\mathcal{Z} : \mathcal{Bord}_2^{\text{def}}(\mathbb{D}) \rightarrow \mathcal{FdHilb},$$

with the usual symmetric monoidal structure.

Given a theory, Davydov, Kong, and Runkel build a **pivotal 2-category**

\mathcal{B}_Z :

- (i) objects: 2-dimensional regions on bordisms
- (ii) 1-morphisms: 1-dimensional line defects
- (iii) 2-morphisms: junction points/local operators
- (iv) adjoints for 1-morphisms: orientation reversal

This is not a classification (yet), since we need to go in the opposite direction.

These defect theories satisfy (A_1) and (A_2) .

So, what if we make them unitary (A_3)?

Surely, they should be “semisimple.”

Douglas and Reutter (2018): a semisimple 2-category

- (i) has adjoints for 1-morphisms.
- (ii) is locally semisimple.
- (iii) is additive.
- (iv) is idempotent complete.

Idea: $\mathcal{B}\mathcal{Z}$ should be (at least, almost) semisimple:

- (i) We already have adjoints for 1-morphisms.
- (ii) Baez (1996): By giving a $\mathcal{F}dHilb$ -category a nice antilinear map $\dagger : \text{hom}(x, y) \rightarrow \text{hom}(y, x)$, we can obtain semisimplicity (“2-Hilbert spaces”).
- (iii) ...
- (iv) We should have local idempotent completeness by (ii).

Further reading:

- (C) Nils Carqueville's *Lecture notes on 2-dimensional defect TQFT*
- (K) Joachim Kock's book *Frobenius Algebras and 2D Topological Quantum field Theories*
- (HV) Chris Heunen and Jamie Vicary's book *Categories for Quantum Theory: An Introduction*

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Thanks! Questions?

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