Quantum Channels and Error Correction

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Math⊗Quantum

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Overview

1. Motivation
2. Quantum Channels
3. Errors as Channels

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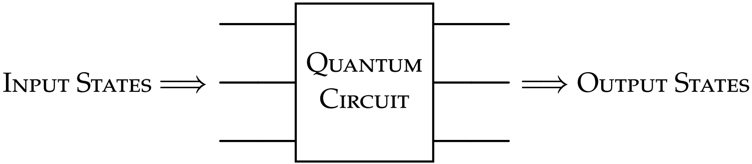
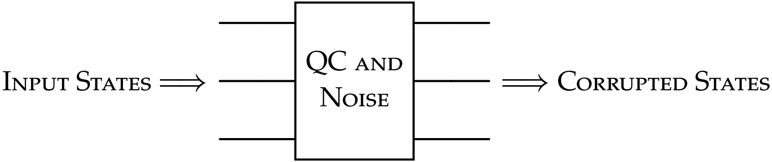
Quantum computation is inherently susceptible to quantum noise–collections of errors caused by quantum mechanical phenomena.


Input States ⟹
Quantum Circuit
⟹ Output States


Input States ⟹
QC and Noise
⟹ Corrupted States

To combat such errors, the discipline of quantum error correction attempts to describe, categorize, and develop recovery processes for quantum noise.

Quantum channels give us a language to mathematically describe the physical process of noise and recovery.

More generally, channels describe how the information of quantum systems transforms.

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To study quantum systems mathematically, we must restrict ourselves via some postulates or axioms.

The state space axiom tells us that we model a quantum system Q using a Hilbert space $\mathscr{H}^Q$ called the state space.

If our system Q consists of a single qubit q, we use the state space $\mathscr{H}^Q = \mathbb{C}^2$. Thus, the mathematics required to understand Q is the linear algebra describing how vectors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

transform.

In particular, the physical states ρ of Q are represented by the 2×2 matrices $\mathbb{M}_2(\mathbb{C})$ which act on the vectors of $\mathcal{H}^Q$.

We specifically require that our states $\rho = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfy

(i) *unit trace*: the sum of the diagonal $a + d = 1$.

(ii) *positivity*: for all vectors $|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$,

$$\langle\psi|\,\rho\,|\psi\rangle = \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \geq 0.$$

We restrict our attention to the unit-trace, positive matrices ρ because we want them to behave like probability distributions.

The multiple systems axiom tells us to use the tensor product to combine state spaces for joint systems.

If system A has state space $\mathcal{H}^A$ and system B has state space $\mathcal{H}^B$, then the joint system AB has state space $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$.

If system A has a single qubit q_1 and system B has a single qubit q_2, then $\mathcal{H}^A = \mathbb{C}^2$ and $\mathcal{H}^B = \mathbb{C}^2$, so the joint state space is

$$\mathcal{H}^{AB} = \mathbb{C}^2 \otimes \mathbb{C}^2.$$

This state space has dimension $2^2 = 4$.

If we instead had n single-qubit systems with qubits $q_1, \ldots, q_n$, then our state space becomes

$$\mathbb{C}^2 \underbrace{\otimes \cdots \otimes}_{n \text{ times}} \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes n}.$$

This state space has dimension 2^n.

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The unitary evolution axiom tells us that if our system Q is closed, i.e., if it does not exchange energy with its environment E, then states ρ transform by a unitary transformation.

If Q is again a single-qubit system with state space $\mathbb{C}^2$, then a unitary transformation is a matrix $U \in \mathbb{M}_2(\mathbb{C})$ whose inverse U^{-1} is its adjoint $U^\dagger$:

$$UU^\dagger = U^\dagger U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

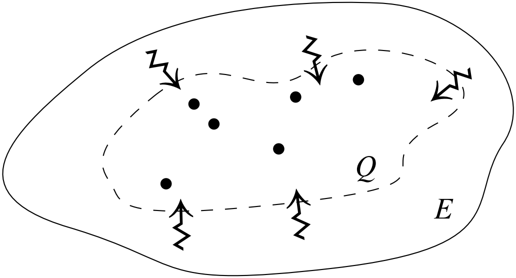
In a sense, this means unitary transformations are the reversible processes on our system Q.

Quantum gates, like those in a quantum circuit, are unitary transformations.

Let Q be a system of n qubits. Then, the corresponding state space is $\mathcal{H}^Q = (\mathbb{C}^2)^{\otimes n}$, and the states are unit-trace, positive matrices ρ in $\mathbb{M}_{2^n}(\mathbb{C})$.

What if Q does exchange energy with its environment E?

In this case, we consider Q to be an open system.


Q
E

Using our method of joining systems, we could instead consider the closed system QE, coupling the n-qubit system Q with its environment E.

Recall that this system QE has state space $\mathcal{H}^Q \otimes \mathcal{H}^E$. Since $\mathcal{H}^Q$ is $(\mathbb{C}^2)^{\otimes n}$, this means

$$\mathcal{H}^{QE} = (\mathbb{C}^2)^{\otimes n} \otimes \mathcal{H}^E.$$

Since our quantum computation occurs in system Q, we effectively only care about the information stored in $\mathcal{H}^Q$.

The partial trace tr_E over the environment is a matrix that acts on the states of $(\mathbb{C}^2)^{\otimes n} \otimes \mathcal{H}^E$.

It is of the form $\mathrm{tr}_E = \mathbb{1} \otimes \mathrm{tr}$, where $\mathbb{1}$ means "do nothing" to the part in $\mathcal{H}^Q$, and tr means "add up the diagonal" of the part in $\mathcal{H}^E$.

The partial trace tr_E gives us a way to restrict our attention to what happened in Q, after working with the joint system QE.

A quantum channel $\mathcal{E}$ is a matrix which takes the states of a joint system to states, even after taking the partial trace over one of the systems.

In the context of QE, this means we can start with a state ρ on $(\mathbb{C}^2)^{\otimes n} \otimes \mathcal{H}^E$, and $\mathcal{E}(\rho)$ is still a state.

Further, we could apply tr_E, and the result $\mathrm{tr}_E(\mathcal{E}(\rho))$ is still a state.

Quantum channels are also called completely positive, trace-preserving (CPTP) maps, precisely because they preserve the properties we required for states, even after taking the partial trace.

Let $\mathcal{E}$ be a quantum channel acting on the states of $\mathcal{H}^Q$. In particular, we want $\mathcal{E}$ to represent noise.

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A theorem of Karl Kraus tells us that for any state ρ, we can write our quantum channel of noise as

$$\mathcal{E}(\rho) = \sum_{k=1}^{r} E_k \rho E_k^{\dagger},$$

where $E_1, \ldots, E_r$ are matrices in $\mathbb{M}_{2^n}(\mathbb{C})$.

Then, we call each of the $E_1, \ldots, E_r$ errors, so the quantum channel of noise $\mathcal{E}$ really consists of small errors which act on our state ρ.

Pick out some space $\mathscr{C} \subseteq \mathscr{H}^Q$. We will call $\mathscr{C}$ the code space, and this is where we will store our information states.

We say that the noise $\mathscr{E}$ on $\mathscr{H}^{Q}$ is correctable if there is a quantum channel $\mathscr{R}$ on $\mathscr{H}^{Q}$ such that for all states ρ on the code space $\mathscr{C}$,

$$\mathscr{R}(\mathscr{E}(\rho)) = \lambda\rho,$$

where λ is some complex scalar.

That is, in the language of channels, correctable means we can find a recovery quantum channel which undoes the effect of the noise $\mathcal{E}$, up to rescaling.

Since $\mathcal{E}$ consists of errors $E_1, \ldots, E_r$, our recovery is also just a collection of corrections $R_1, \ldots, R_r$ in $\mathbb{M}_{2^n}(\mathbb{C})$, each of which should counteract the effects of the errors.

We can thus write the effect of our recovery $\mathcal{R}$ on a state ρ as

$$\mathcal{R}(\rho) = \sum_{\ell=1}^{r} R_\ell \rho R_\ell^\dagger,$$

where $R_1, \ldots, R_r$ are the corrections.

The framework of quantum channels allows us to translate between the notions of noise and recovery on a full system and errors and corrections on parts of the system.

When studying quantum error correcting codes, researchers precisely try to find code spaces $\mathcal{C}$ on which they can construct a recovery $\mathcal{R}$ for the quantum channel of noise $\mathcal{E}$.

In practice, this means one must work with the individual errors $E_1, \ldots, E_r$, looking to find a way to build the quantum channel $\mathcal{R}$ out of corrections $R_1, \ldots, R_r$.

Further reading:

(L) Felix Leditzky's notes *Quantum Channels* (link).

(NC) Nielsen and Chuang's book *Quantum Computation and Quantum Information*, sections §8 and §10.

(KLM) Kaye, Laflamme, and Mosca's book *An Introduction to Quantum Computing*, section §10.

Questions?

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