Winter Spaces and the Stabilizer Formalism

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Background

Quantum systems can be modeled as Hilbert spaces. A quantum state can be described as a density matrix $\rho \in \mathbb{B}(\mathcal{H})$, which is a positive semidefinite linear operator on the Hilbert space $\mathcal{H}$.

The evolution of quantum systems is modeled via quantum channels. This can be understood as the noise introduced to the system. A quantum channel $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ is a completely positive and state-preserving map sending density matrices to density matrices.

Given that any channel $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$, the Stinespring dilation theorem guarantees the existence of a Kraus representation.

$$\Phi(\rho) = \sum_{a=1}^{n} E_a \rho E_a^\dagger.$$

Kraus representation is unique up to a unitary matrix.

Knill-Laflamme Subspace Condition

Given a noise model (channel) $\mathcal{E}$, to do quantum error correction, we aim to find a recovery operation $\mathcal{R}$ such that given any state in an appropriate subspace,

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \rho.$$

However, not all errors are correctable. For example, the error of deleting all information is not a reversible operation. The following theorem gives a necessary and sufficient condition for all correctable errors.

Theorem (Knill-Laflamme Subspace Condition)**.** *Given a channel $\mathcal{E} : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ and its Kraus operators $\{E_a\}_{a\in\Lambda}$. $\mathcal{E}$ is correctable (the recovery map exists) if and only if there exists a subspace $\mathcal{C} \subseteq \mathcal{H}$ and an orthogonal projection $P : \mathcal{H} \to \mathcal{C}$ such that*

$$P E_a^\dagger E_b P = \lambda_{ab} P.$$

This is a condition on all products of Kraus operators, which is rather inconvenient to check. Let the winter graph (non-commutative graph) be defined as

$$\mathcal{V} = \operatorname{span}\left\{ E_a^\dagger E_b : a, b \in \Lambda \right\}.$$

The above conditions can be rewritten as

$$P\mathcal{V}P = \mathbb{C}P \quad \text{or} \quad \dim P\mathcal{V}P = 1.$$

Classical Stabilizer Formalism

The stabilizer formalism provides a concrete code space $\mathcal{C}$ for a special class of error models. Given an Abelian subgroup $G \subseteq P_n$ where P_n denotes the Pauli group on n qubits. The error is correctable if and only if

$$E_a^\dagger E_b \in \operatorname{span}\{P_n \backslash N(S) \cup S\} = \operatorname{span}\{P_n \backslash \mathcal{Z}(S) \cup S\} \cong P_s,$$

where S is the stabilizer of G and $\mathcal{Z}(S)$ is the center of S.

Winter Graph Framework for Stabilizer Formalism

In operator theory, Winter graphs are exactly operator systems. That is, $\mathcal{V} \subseteq \mathbb{B}(\mathcal{H})$ where $\mathcal{V}$ is unital and self-adjoint.

A special class of Winter graphs can be generated by the unitary representation of the error model. Given a compact group G and a unitary representation $\pi : G \to \mathcal{U}(n)$, the Winter graph can be written as

$$\mathcal{V}_{M_0} = \operatorname{span}\{\pi(g) M_0 \pi(g)^* : g \in G\}.$$

To recover the stabilizer formalism, consider the stabilizer subgroup S of some abelian subgroup of Pauli group P_n. Taking the Winter graph generated by the trivial representation $\pi : S \to \mathcal{U}(n)$ where every element in S is sent to itself, it generates a class of Winter graphs

$$\mathcal{V}_{M_0} = \operatorname{span}\{g M_0 g^* : g \in S\}.$$

Not all M_0 give rise to an operator system. However,

$$\operatorname{span}\left\{\mathcal{V}_{M_0} : M_0 \text{ makes } \mathcal{V}_{M_0} \text{ into an operator system}\right\} = \operatorname{span}\left\{P_n \backslash \mathcal{Z}(S) \cup I^{\otimes n}\right\}$$

recovers from stabilizer formalism.

Operator Quantum Error Correction (OQEC)

A natural generalization of the stabilizer formalism is the OQEC framework. Consider the decomposition of the Hilbert space

$$\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \oplus \mathcal{K}.$$

$\mathcal{H}^A$ is called a noiseless subsystem if for any $\sigma^A \in \mathcal{H}^A$ and $\sigma^B \in \mathcal{H}^B$,

$$\mathcal{E}(\sigma^A \otimes \sigma^B) = \sigma^A \otimes \tau^B$$

for some $\tau^B \in \mathcal{H}^B$.

Theorem. *$\mathcal{H}^A$ is a noiseless subsystem if and only if there exists a projection $P : \mathcal{H} \to \mathcal{H}^A \otimes \mathcal{H}^B$ such that*

$$P E_a^\dagger E_b P = (I \otimes B_{ab}) P$$

Stabilizer Formalism for OQEC

Let $S = \langle Z_1, \dots Z_s \rangle$ be the stabilizer and $N(S) = \langle i, Z_1, \dots, Z_s, X_{s+1}, Z_{s+1}, \dots, X_n, Z_n \rangle$ be its normalizer. Consider the gauge group $\mathcal{G} = \langle i, Z_1, \dots, Z_s, X_{s+1}, Z_{s+1}, \dots, X_{s+r}, Z_{s+r} \rangle$. This system has s stabilizer qubits, r gauge qubits, and $k = n - s - r$ logical qubits that absorb errors.

The stabilizer formalism for OQEC takes advantage of the gauge qubits on the noiseless A-system and gets the code space

$$E_a^\dagger E_b \in \operatorname{span}\{P_n \backslash \mathcal{Z}(S) \cup \mathcal{G}\} \cong P_{s+r}.$$

This exactly corresponds to the fact that the A system is noiseless and thus all the stabilizer and gauge qubits are saved.

A Winter Space Approach

In [3], it is shown that the Winter space framework is capable of recovering the stabilizer formalism. This project expands on the previous results by checking the same for qudits and for the gauge qudits under OQEC framework.

We begin by defining the generalized Pauli group. For the convenience of notation, we will write X and Z as of the generating matrix for the shift and phase operators, defined as

$$X|j\rangle = |j+1\rangle \quad \text{and} \quad Z|j\rangle = \omega^j |j\rangle$$

where $\omega = e^{2\pi i/d}$ is the d-th root of unity.

For the stabilizer subgroup for qudits given by

$$S = \langle Z_1, Z_2, \dots, Z_s \rangle = \left\{ Z^{k_1} \otimes Z^{k_2} \otimes \cdots \otimes Z^{k_s} : k_i = 1, \dots, d-1 \right\}$$

and the gauge group for qudits given by

$$\mathcal{G} = \langle Z_1, Z_2, \dots, Z_s, X_{s+1}, Z_{s+1}, \dots X_{s+r}, Z_{s+r} \rangle$$

we are able to recover the stabilizer formalism as

$$\operatorname{span}\left\{\mathcal{V}_{M_0} : M_0 \text{ makes } \mathcal{V}_{M_0} \text{ into an operator system}\right\} = \operatorname{span}\left\{P_n \backslash \mathcal{Z}(\mathcal{G}) \cup I^{\otimes n}\right\}.$$

Further Generalization and Future Works

We will generalize the Winter graph framework to other error correction codes and frameworks:

Clifford Code: generalization of stabilizer codes for non-commutative error models, such as error model with index $\mathbb{Z}_2 \times D_8$ first.

A Winter space approach to stabilizer formalism for operator algebra quantum error correction: Generalization of our framework to the regime of operator algebra quantum error correction recently introduced by Kribs et al.

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