CUBICAL COMPLEXES AND 3-MANIFOLDS

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1. Introduction to Cube Complexes

A *cube complex* has the data of a set of Euclidean unit cubes $\mathcal{C}$ together with a set of isometries $\mathcal{F}$ between faces of cubes in $\mathcal{C}$. We define a cube complex $X = \mathcal{C}/\mathcal{F}$ by identifying the domains of these maps with their images in $\coprod_{C\in\mathcal{C}} C$. The faces of a cube complex are the images of faces under the quotient map $\coprod_{C\in\mathcal{C}} C \to X$. The *dimension* of X will be the largest n so that there is an n-cube in $\mathcal{C}$.

1.1. NPC and CAT(0) cube complexes.

We move toward non-positively curved (NPC) and CAT(0) cube complexes by investigating the local information of a cube complex.

Definition 1.1 (Link Complex)**.** For $v \in X$ a vertex, we can define an abstract simplicial complex called the *link complex*, denoted $\mathrm{lk}(x)$. The vertices of the link complex are *local edges*: subintervals of edges of length $1/3$, one of whose endpoints is v. The simplices of the link complex are sets of local edges contained in a common cube.

Definition 1.2 (Flag Complex)**.** A flag complex is an abstract simplicial complex satisfying the following conditions:

(i) The 1-skeleton is a simple graph;

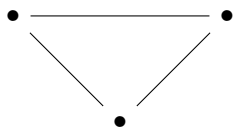
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(ii) For every complete graph in the 1-skeleton, its vertices span a simplex.

Definition 1.3 (NPC Cube Complex)**.** A cube complex X is an NPC cube complex if it satisfies Gromov's link condition: every link complex in X is a flag complex.

Example 1.4. One important example of a cube complex that does not satisfy the link condition is the 'half cube'. The half cube is the union of any three faces in a 3-cube which share a vertex. The link complex associated to this common vertex is the following graph:



This is not a flag complex since there is no simplex containing all three edges, and thus the half cube is not an NPC cube complex.

Definition 1.5 (CAT(0) Cube Complex)**.** An NPC cube complex X is a CAT(0) cube complex if it is simply connected.

Remark 1.6. The difference between NPC and CAT(0) cube complexes is an instance of local-to-global phenomenona. The Cartan-Hadamard theorem tells us that the universal cover of a connected NPC complete metric space is a CAT(0) space. An NPC cube complex is *locally* CAT(0), and if it is simply connected (and thus a CAT(0) cube complex), then it is its own covering space, meaning that it is *globally* CAT(0).

Proposition 1.7. *An NPC cube complex is simply connected if and only if its 2-skeleton is.*

Proof. The statement follows from the fact that an n-dimensional cube complex is a CW complex and the path components and fundamental group of CW complex depend only on its 2-skeleton. □

Example 1.8 (Graphs)**.** Any 1-dimensional cube complex is simply a graph. The link of a vertex in a graph has no edges, so every link satisfies the flag condition, and every graph is an NPC cube complex. The universal cover of a graph is a tree, which is additionally a CAT(0) cube complex.

Example 1.9 (Tori)**.** A torus is constructed from a square, identifying opposite edges, and is thus a cube complex. It has only one vertex, and its link is a cycle of length 4, so it is an NPC complex.

Example 1.10 (Surfaces)**.** Generalizing the previous example, an orientable surface Σ_g of genus $g > 1$ is an NPC cube complex. To realize the surface as a cube complex, we must first construct a $4g$-gon from cubes. This can be done by adding vertices at the center and the barycenters of the edges and connecting the barycenters to the center by edges. The $4g$-gon can then be folded to produce Σ_g.

1.2. **Hyperplanes.** By a *midcube* of an n-dimensional cube we will mean an $n-1$-dimensional cube that intersects the barycenter and is parallel to one of its faces.

Definition 1.11 (Hyperplane)**.** Consider the equivalence relation generated by the relation $e_1 \sim e_2$ if e_1 and e_2 are opposite edges of a square. Now, given an equivalence class $[e]$, the hyperplane dual to $[e]$ is the collection of midcubes that intersect edges in $[e]$.

Theorem 1.12. *If X is a $CAT(0)$ cube complex, then the following hold:*

(i) Every hyperplane in X is embedded.
(ii) Every hyperplane of X separates X into precisely two halfspaces.
(iii) Every hyperplane is a $CAT(0)$ cube complex.
(iv) Every collection of pairwise intersecting hyperplanes intersects.

Example 1.13 (Hyperplanes of Graphs)**.** The midcube of a 1-cube is simply the midpoint of the edge. Since a graph contains no squares, this midcube is the whole hyperplane.

Example 1.14 (Hyperplanes of a Torus)**.** Each hyperplane of a torus is a closed curve. In the universal cover of a torus, which is a Euclidean plane tiled with squares, the hyperplanes are horizontal or vertical lines.

2. Building CAT(0) Cube Complexes

In this section, we will go over a strategy to build CAT(0) cube complexes. This will require that we first introduce pocsets and ultrafilters.

2.1. Pocsets and ultrafilters.

Definition 2.1 (Pocsets)**.** A pocset Σ is a partially ordered set with a complementation. Here, complementation means that there is an order reversing involution $A \mapsto A^*$ for all $A \in \Sigma$ such that $A \neq A^*$ and A, A^* are always incomparable by the pocset's relation. Furthermore, complementation is order reversing: if $A > B$, then $B^* > A^*$.

The above allow us to consider intervals $[A, B] = \{C | A < C < B\}$. If all intervals are finite, then we say that the pocset is *locally finite*. From here we will take all our pocsets to be locally finite.

Definition 2.2 (Nested and Transverse Elements)**.** We say that two elements A, B of Σ are nested if and only if one of the following holds: $A < B$, $A < B^*$, $A^* < B$, or $A^* < B^*$. Otherwise, we say they are transverse.

Definition 2.3 (Width of Pocset)**.** The width of a pocset Σ is defined as the size of its maximal transverse subset.

The motivating example of a pocset is the *halfspaces of a cube complex*, which we will denote $\mathcal{H}(X)$. We consider the set of halfspaces with the subset relation. Complementation of a halfspace is given by taking the other halfspace which is bound by same hyperplane. This set will have a structure of a pocset and it will have width equal to the dimension of the complex.

This example is important because our goal will be to construct a CAT(0) cube complex from an arbitrary pocset so that applying the construction to the pocset of half spaces of a CAT(0) cube complex recovers the original complex.

Definition 2.4 (Ultrafilters)**.** An ultrafilter α on a pocset Σ is a subset of Σ which satisfies the following:

- Choice: for all pairs $\{A, A^*\}$ in Σ, precisely one of them is in α
- Consistency: $A \in \alpha$ and $A < B$ implies $B \in \alpha$

Definition 2.5 (Descending Chain Condition)**.** An ultrafilter α is said to satisfy the Descending Chain Condition if every descending chain of elements terminates.

To better understand ultrafilters, we will consider the pocset of half spaces of a tree. In a tree, each edge exactly partitions the entire tree into two complementary regions separated by the edge's midpoint. Then, the ultrafilter is composed by choosing exactly one of the two halfspaces per edge across all edges such that they satisfy the consistency condition. A very useful representation of the ultrafilter consists in fixing arrows on all edges and choosing their directions such that they point towards the chosen halfspace. In particular, this consistency condition implies that, once a halfpsace is chosen at a particular edge, then all arrows in its discarded halfspace must point towards the chosen halfspace.

A particular choice of ultrafilter α_v can be constructed by choosing a vertex v in the tree and choosing all the halfspaces such that their arrows point towards that vertex. These ultrafilters satisfy the Descending Chain Condition (since all the chains must terminate at one of the halfspaces formed on edges touching the vertex). Furthermore, we can show that any ultrafilter that satisfies the Descending Chain Condition corresponds to a vertex ultrafilter α_v. Indeed, otherwise, it would mean that some descending chains terminate around one other vertex v^*. Now, we need to consider the shortest path γ between v and v^*. Since v and v^* are terminal vertices, the edges that are both on γ and directly neighboring v and v^* must point towards v and v^* respectively. However, this means that there is a vertex $c \in \gamma$ whose neighborhood locally looks like:

$$v \ldots \longleftarrow c \longrightarrow \ldots v^*$$

This scenario however breaks the consistency condition of ultrafilters. The case where chains terminate around 3 or more vertices can be dealt in the same way.

Now, let's consider ultrafilters which do not satisfy the Descending Chain Condition in a tree. This means that, for each vertex v in the tree, there must be exactly one edge in its neighborhood which points away from it. Indeed, consistency condition requires at most one edge pointing away from any vertex, and since ultrafilter does not satisfy DCC, there must be at least one edge pointing away from any vertex. Therefore, ultrafilters which do not satisfy DCC exactly determine *rays* whose closure correspond to boundary points of the tree. In more general cube complexes, these points represent the Roller boundary.

2.2. **Cubulations.** We will now see how to construct a cube complex $X = X(\Sigma)$ from a finite width pocset Σ. The vertex set $X(\Sigma)^0$ will be the set of DCC ultrafilters. We have seen that these are in bijection for trees and the following proposition shows this is more general.

Proposition 2.6. *Every DCC ultrafilter in the pocset of halfspaces $\mathcal{H}(X)$ associated to a $CAT(0)$ cube complex is of the form $\alpha_v = \{\mathfrak{h} \in \mathcal{H}(X) : v \in \mathfrak{h}\}$.*

Proof. Consider a DCC ultrafilter α on $\mathcal{H}$. By Zorn's lemma, every chain in this ultrafilter contains a minimal element $\mathfrak{h}_i$. Furthermore, the pairwise intersections between

minimal elements is nonempty. Otherwise there exist $\mathfrak{h}_i, \mathfrak{h}_j$ so that both are minimal elements and $\mathfrak{h}_i \cap \mathfrak{h}_j = \emptyset$. This implies that $\mathfrak{h}_i \subset \mathfrak{h}_j^*$. However, this means that, by the ultrafilter's consistency condition, $\mathfrak{h}_j^* \in \alpha$ which contradicts the choice condition as $\mathfrak{h}_j \in \alpha$ as well. Since the pairwise intersection of minimal elements is nonempty, the intersection of all minimal elements is nonempty. This intersection contains a vertex because halfspaces (and the intersection of halfspaces) are themselves cube complexes and any nonempty cube complex contains a vertex.

Finally, we will show that this vertex is unique. Let v, w be two different vertices which lie in the intersection of all halfspaces that belong to α. Since $v \neq w$, there exists a hyperplane which separates them, with its two halfspaces $\mathfrak{h}$ and $\mathfrak{h}^*$ such that $v \in \mathfrak{h}$ and $w \in \mathfrak{h}^*$. Since α is a ultrafilter, it must contain exactly one of $\mathfrak{h}$ or $\mathfrak{h}^*$. If $\mathfrak{h} \in \alpha$, then $w \notin \mathfrak{h}$, which is a contradiction. Similarly, if $\mathfrak{h}^* \in \alpha$, then $v \notin \mathfrak{h}^*$ which is again a contradiction. □

Now we will declare that two vertices $v, w \in X(\Sigma)^0$ are joined by an edge if $|v \Delta w| = 2$, i.e. if there exists $A \in \Sigma$ such that $w = (v - \{A\}) \cup \{A^*\}$. When A is minimal in v, then we use the notation $(v; A) = (v - \{A\}) \cup \{A^*\}$, and $(v; A, B) = ((v; A); B)$ when B is minimal in $(v; A)$.

Proposition 2.7. *Let $v \in X(\Sigma)^0$. For $A \in v$, $(v - \{A\}) \cup \{A^*\}$ is an ultrafilter if and only if A is minimal in v. If this is the case then $(v - \{A\}) \cup \{A^*\} \in X(\Sigma)^0$.*

Proof. Let v be an ultrafilter. Let $A \in v$ and $(v - \{A\}) \cup \{A^*\}$ be an ultrafilter. If A was not minimal, then there exists $B \in v$ so that $B < A$. Now, $(v - \{A\}) \cup \{A^*\}$ contains both A^* and B as halfspaces. However, by consistency condition, $B \in v$ and $B < A$ implies $A \in v$. This contradicts the choice condition.

Let v be an ultrafilter and A one of its minimal elements. Now, $(v - \{A\}) \cup \{A^*\}$ still satisfies the choice condition since A and A^* are swapped. Now, we must check its consistency. Let $B \in (v - \{A\}) \cup \{A^*\}$ and $B < C$ for some halfspace C. We must show that $C \in (v - \{A\}) \cup \{A^*\}$ We have two cases:

- $B \neq A^*$. Then, $B \in v$. Therefore, by consistency of v, $C \in v$. Furthermore, $C \neq A$ since that would contradict minimality of A in v.
- $B = A^*$. Then $A^* < C$ if and only if $C^* < A$ implies $C \neq A$. Furthermore, since v is an ultrafilter, either $C \in v$ or $C^* \in v$. However, $C^* \in v$ would break minimality of A, unless $C^* = A$ if and only if $C = A^*$, and therefore $C \in (v - \{A\}) \cup \{A^*\}$. Otherwise, $C \in v$, and since $C \neq A$, $C \in (v - \{A\}) \cup \{A^*\}$

Lastly we wish to argue that $(v - \{A\}) \cup \{A^*\}$ is DCC. Suppose we have a descending chain in v, $(A_n)_{n \in \mathbb{N}}$, which does not contain A and for which A is the unique lower bound. Now let L_m be the set of lower bounds in v of $(A_n)_{n \leq m}$. We note that $L_{m+1} \subseteq L_m$ and $\bigcap L_m$ is the set of lower bounds of the sequence. By assumption we have $\bigcap L_m = \{A\}$, but local finiteness tells us that our set is discrete, so this implies there is an m so that $L_m = A$. This can only be the case if $A_m = A$ otherwise we would have $\{A, A_m\} \subseteq L_m$. By contradiction we find conclude that there are no descending chains in $(v - \{A\})$ with no lower bound. In turn, we find there are no descending chains in $(v - \{A\}) \cup \{A^*\}$ with no lower bound (since any descending chain containing A^* has A^* as a lower bound). We have thus proven $(v - \{A\}) \cup \{A^*\}$ is DCC. □

Again we argue that our choice makes sense based on what we know about pocsets of cube complexes.

Proposition 2.8. *For X a CAT(0) complex with $u, v \in X^0$ and e an edge $u \to v$ let $\hat{\mathfrak{h}}$ be the hyperplane dual to $[e]$ and $\mathfrak{h}$ the halfspace containing u and bounded by this hyperplane. Then $(\alpha_u; \mathfrak{h}) = \alpha_v$.*

Proof. The proposition follows from the fact that $\hat{\mathfrak{h}}$ is the unique hyperplane separating u, v. Note that any hyperplane separating u and v must intersect e and therefore is the hyperplane dual to $[e]$.

Now we find that $\mathfrak{h}$ is minimal in α_u, otherwise we would have another hyperplane separating u and v. In turn, every halfspace in $\alpha_u \backslash \mathfrak{h}$ contains v and thus every halfspace in $(\alpha_u; \mathfrak{h})$ contains v. But we have seen that there is a unique vertex contained in every halfspace of a DCC ultrafilter so $(\alpha_u; \mathfrak{h}) = \alpha_v$. □

We also show that the resulting 1-skeleton is path connected.

Proposition 2.9. *If Σ has finite width, then any two DCC ultrafilters are joined by a finite path.*

Proof. Let $v, w \in X^0(\Sigma)$. We need to show that there is an $n \in \mathbb{N}$ such that $w = (v; A_1, \ldots, A_n)$ with $A_i \in v$ and $A_i^* \in w$. We will argue by contradiction. Let's considered the ordered set $\mathcal{S} = \{A_i \in v \mid A_i^* \in w\}$, and let's assume it's infinite. Since v and w are ultrafilters, we can show that for $A, B \in \mathcal{S}$, if A, B are incomparable by inclusion, then they are transverse: Indeed, $A < B$ is not satisfied since they are incomparable, $A < B^*$ implies $B^* \in v$, which is a contradiction to v's choice condition, $A^* < B$ implies $B \in w$, which is also a contradiction to w's choice condition, and $A^* < B^*$ if and only if $B < A$, but A and B are not comparable.

Thus any two elements of $\mathcal{S}$ are either comparable or transverse. Using the infinite Ramsey Theorem (applied to a graph where vertices are the A_i's and edges $[A_i, A_j]$ are colored one of 2 different colors depending on whether A_i and A_j are transverse or comparable), we can show that there is an infinite subset M of that graph in which all edges $[A_i, A_j] \in M^1$ have the same color (i.e. in which the halfspace are either all pairwise transverse or pairwise comparable). However, from finite width, they cannot be all pairwise transverse. Hence, they must be all pairwise comparable, which means that they form a nested sequence. Then, either $A_1 < A_2 < A_3 < \ldots$ or $A_1 > A_2 > A_3 > \ldots$. In the first case, we can consider $A_1^* > A_2^* > A_3^* > \ldots$ which forms a sequence of halfpsaces in w. However, this sequence does not satisfy the Descending Chain Condition, which is a contradiction. In the second case, the sequence of A_i's also does not satisfy the DCC in v, and is a contradiction. Hence, the cardinality of $\mathcal{S}$ is finite, and there is a finite path between v and w. □

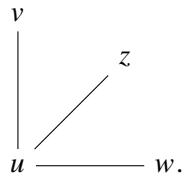
Finally we will complete the construction of the cubulation.

Definition 2.10 (Cubulation)**.** Let Σ be a finite width pocset. We define its cubulation, $X(\Sigma)$. Let $X(\Sigma)^0$ be the set of DCC ultrafilters and define $X(\Sigma)^1$ so that two DCC ultrafilters $v, w \in X(\Sigma)^0$ are joined by an edge if $|v \Delta w| = 2$. Then for $n \geq 2$ let $X(\Sigma)^n$ be correspond to the boundaries of the n-cube which appear in $X(\Sigma)^{n-1}$.

Proposition 2.11. *The cubulation of a finite width pocset is a finite dimensional CAT*(0) *cube complex.*

Proof. We give an idea of the argument that the cubulation is NPC. A link complex is a flag complex if its 1-skeleton is simple and if every complete graph in the 1-skeleton corresponds to a simplex. The first condition is satisfied by because we add exactly one square each time its boundary appears and we do not glue any squares to themselves.

Now, for the second condition, the cases where we have complete graphs with $n = 1, 2$ vertices are trivially satisfied. We will show the case when $n = 3$ and give an idea of why it can be generalized. A triangle in the link complex of a vertex u corresponds to a half cube. To show our triangle corresponds to a simplex we want our cubulation to contain the entire cube, and since our cubulation depends only on the 1-skeleton, it suffices to show that we can 'complete the cube' inside our cubulation by showing it must contain all its edges and vertices. But note that the 1-skeleton of the half cube is only missing the vertex opposite to u' and the edges containing it. Consider the following subcomplex of the half cube:



Let $\{A, B, C\} \subseteq u$ such that $v = (u; A)$, $w = (u; B)$ and $z = (u; C)$. We know that B is minimal in $(u; A)$ since the half cube contains the face with u, v, z and it is enough to show that C is minimal in $((u; A); B)$. If this is the case, then we have by symmetric argument $u' = (((u; A); B); C) = (((u; E); F); G)$ where (E, F, G) is any permutation of (A, B, C). Supposing C is not minimal we must have $A^* < C$ or $B^* < C$, but this is a contradiction because C is minimal in $(u; A)$ and $(u; B)$.

Now for cases $n = 4$ the $n = 3$ case implies that every triangle in our link complex is filled and this means our cubical complex has all the 3-cubes containing the vertex u'. Then we can just repeat the argument we did above. The subsequent cases will proceed by a similar argument.

□

To show that the cubulation is actually a CAT(0) cube complex, one still has to show that the fundamental group of the 2-skeleton is trivial; see [MS14].

Corollary 2.12. *Given a CAT*(0) *cube complex X,* $X(\mathcal{H}(X)) = X$.

Proof. We have already shown that $X(\mathcal{H}(X))^1 = X^1$. Since CAT(0) cube complexes are simply connected, X^2 must contain all the squares which have boundaries in X^1. In turn, the link condition says that for $n \geq 3$, X^{n+1} must contain all the cubes which have boundaries in X^n. □

Proposition 2.13. *The dimension of* $X(\Sigma)$ *is the width of* Σ.

3. Groups Actions on Cube Complexes

In the previous section we showed how we can build CAT(0) cube complexes from pocsets. This section will demonstrate how this allows us to build group actions on CAT(0) cube complexes.

First we define *spaces with walls*. This is a pair of sets (S, Σ) where Σ is a collection of subsets of S which is closed under complementation. Then, Σ is made into a pocset with the usual subset relation.

Example 3.1. The pocset of halfspaces $\mathcal{H}(X)$ of a cube complex is an example of a space with walls.

Example 3.2. We can consider the plane $\mathbb{R}^2$ and the subsets $\{(x, y) \mid x \leq n\}, \{(x, y) \mid y \leq n\}$ for $n \in \mathbb{Z}$ and their complements. This is a space with walls and Σ is a finite width pocset. The resulting CAT(0) cube complex is the tilling of $\mathbb{R}^2$ by squares.

Example 3.3. We again consider the plane $\mathbb{R}^2$ and consider its tesselation by equilateral triangles. Consider the infinite straight lines in this tessalation and let Σ consist of the halfspaces divided by these lines. This is again a space with walls and Σ is a finite width pocset. This time the resulting CAT(0) cube complex is the tilling of $\mathbb{R}^3$ by cubes.

Definition 3.4 (Group Acting on Pocset)**.** Let Σ be a finite width pocset and G a group. We say that G acts on Σ if we have a group action $G \curvearrowright \Sigma$ and for $g \in G$ and $A < B \in \Sigma$ then $gA < gB$ and $gA = (gA^*)^*$.

Theorem 3.5. *If G acts on a finite width pocset Σ then G acts on $X(\Sigma)$.*

Proof. We will show that we have an action on $X(\Sigma)^1$ from which the action on $X(\Sigma)$ can be produced via cellular isometries. Since G acts on the pocset Σ, we have that each element $g \in G$ defines an automorphism: $g : \Sigma \to \Sigma$ with $A < B$ if and only if $gA < gB$ and $g(A^*) = (gA)^*$. Now, let v be a DCC ultrafilter, we define $g \cdot v = \{gA | A \in v\}$. We show that $g \cdot v$ is again a DCC ultrafilter:

- Choice: Since v contains one of $\{A, A^*\}$, $g \cdot v$ contains one of $\{gA, g(A^*) = (gA)^*\}$.
- Consistency: Assume $B \in g \cdot v$ and $B < C$. Then $B = gA$ for some $A \in v$. Applying $g^{-1} \in G$, we get that $A = g^{-1}gA = g^{-1}B < g^{-1}C$. Therefore, by consistency of v, $g^{-1}C \in v$. As such, $C = gg^{-1}C \in g \cdot v$, and $g \cdot v$ satisfies consistency condition.

Now we check the DCC condition. Suppose there were an infinite descending chain in $g \cdot v$:

$$B_1 > B_2 > B_3 > \cdots .$$

Each B_i has the form $B_i = gA_i$ with $A_i \in v$. Applying g^{-1}, we obtain

$$A_1 > A_2 > A_3 > \cdots ,$$

an infinite descending chain in v, contradicting that v satisfies DCC. Hence $g \cdot v$ is again a vertex of $X(\Sigma)$. It remains to check that edges are preserved. If v, w are adjacent, then they differ on exactly one complementary pair, say $w = (v; A)$. That means $A \in v$ and $A^* \in w$, while v and w agree on every other complementary pair. Applying g, the

ultrafilters $g \cdot v$ and $g \cdot w$ differ exactly on the pair $\{gA, (gA)^*\}$. Therefore $g \cdot w = (g \cdot v; gA)$ □

Definition 3.6 (Codimension-1 Subgroup)**.** Let G be a group that is finitely presented. We say that H is a subgroup of G of codimension 1 if the coset graph G/H has more than one end. This means that the H separates the Cayley graph of G into two. In particular, there is an $R \in \mathbb{R}$ such that the R-neighborhood of H separates the Cayley graph into two deep components (here deep means that it is not contained in any finite neighborhood of H).

If we choose one of these components $A \subset G$, then the G translates of A and $A^* = G \backslash A$ form walls $\Sigma = \{gA, gA^* | g \in G\}$ on the set G. This gives us the structure of a space with walls and G acting on its pocset and associated cubulation.

4. Applications to 3-Manifolds

One of the most noteworthy uses of CAT(0) cube complexes is in the theory of 3-manifolds. Recall that a *smooth n-manifold* is a (second countable, Hausdorff) topological space M that admits a smooth atlas of charts, i.e., a collection $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in \mathcal{A}}$ of maps such that for all indices $\alpha, \beta \in \mathcal{A}$,

(i) the U_α form an open cover of M.
(ii) φ_α is a homeomorphism onto its image.
(iii) either $U_\alpha \cap U_\beta$ are disjoint or the composite

$$\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is smooth.

Hereafter, "manifold" will mean "smooth n-manifold." At each point $p \in M$, there is an $\mathbb{R}$-vector space T_pM called the *tangent space* at p.

Definition 4.1 (Riemannian Manifold)**.** A Riemannian metric on a manifold M is a choice of inner product[1]

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

for each point $p \in M$. The pair (M, g) is a Riemannian manifold.

Given a two-dimensional subspace $S \subseteq T_pM$, the *sectional curvature* of M at p is a real quantity $K(S)$ defined using the Riemannian structure of (M, g). If $K(S)$ is some constant κ for all choices of $S \subseteq T_pM$ and $p \in M$, then we say M has constant sectional curvature κ.

4.1. Hyperbolic manifolds and groups.

Definition 4.2 (Hyperbolic Manifold)**.** A hyperbolic manifold is a (usually connected) Riemannian manifold (M, g) with constant sectional curvature -1 such that M is Cauchy

[1]We often also have smoothness conditions on g.

complete with respect to the metric[2]

$$d_g(x,y) = \inf_{\substack{\gamma \text{ is admissible} \\ \gamma(0)=x \text{ and } \gamma(1)=y}} \left\{ \text{length}(\gamma) = \int_0^1 \|\gamma'(t)\| \, \mathrm{d}t \right\}.$$

Example 4.3. The simplest example of a hyperbolic n-manifold is $\mathbb{H}^n$. As we did with $\mathbb{H}^2$, we let $\mathbb{H}^n$ be the upper half of $\mathbb{R}^n$ with the rescaled metric.

Example 4.4. Let K be a knot in the 3-sphere. In other words, K is an embedding $\mathbb{S}^1 \to \mathbb{S}^3$. Suppose that K cannot

(i) be placed on a torus in $\mathbb{S}^3$.
(ii) arise as a nontrivial embedding of $\mathbb{S}^1$ into a small neighborhood $N(K') \simeq \mathbb{S}^1 \times \mathbb{D}^2$ of another knot K'.[3]

Then, Thurston showed that $\mathbb{S}^3 \setminus K$ is a hyperbolic 3-manifold [Thu82].

Now, let (M, g) be hyperbolic and $\pi : \widetilde{M} \to M$ be a covering map.

Proposition 4.5. *The covering space $\widetilde{M}$ is a topological n-manifold. Moreover, it admits a unique smooth structure so that π is a smooth covering, i.e., a smooth surjection so that every connected neighborhood in M is evenly covered by diffeomorphisms.*

Proof. See, for example, [Lee12]. □

It turns out that we can do better than smooth.

Proposition 4.6. *The covering manifold $\widetilde{M}$ can be given a hyperbolic structure.*

Sketch. We can define an inner product

$$\widetilde{g}_p : T_p\widetilde{M} \times T_p\widetilde{M} \to \mathbb{R}$$

by

$$\widetilde{g}_p(x,y) = g_{\pi(p)}(\mathrm{d}\pi_p(x), \mathrm{d}\pi_p(y)),$$

where $\mathrm{d}\pi_p : T_p\widetilde{M} \xrightarrow{\simeq} T_{\pi(p)}M$ is the differential. The map π is a local isometry, so it in particular preserves the sectional curvature $K(S)$ for any $S \subseteq T_p\widetilde{M}$. Yet, M is hyperbolic, so $K(S) = -1$ for all S and $p \in \widetilde{M}$. □

Now, note that there is, up to isometry, precisely one simply connected hyperbolic n-manifold: $\mathbb{H}^n$. This follows from the Cartan-Hadamard theorem.

Lemma 4.7. *The universal cover of a hyperbolic n-manifold M is (isometric to) $\mathbb{H}^n$.*

Proof. The universal cover $\pi : \mathcal{M} \to M$ is simply connected, by definition, and we now know that it has the structure of a hyperbolic manifold. Thus, $\mathcal{M}$ is $\mathbb{H}^n$. □

Corollary 4.8. *We can write M as a quotient $M = G\backslash\mathbb{H}^n$, where G is a discrete group of isometries on $\mathbb{H}^n$.*

[2]For γ to be admissible means it is a piecewise smooth map $\gamma : [0,1] \to M$ such that $\gamma'(t)$, wherever defined, is nonzero in $T_{\gamma(t)}M$. The norm in $T_{\gamma(t)}M$ comes from the inner product $g_{\gamma(t)}$.

[3]Here, the adjective "nontrivial" means the embedding is not isotopic to K'.

Proof. By an application of the chain rule, the group G of deck transformations $\Gamma(\mathbb{H}^n \to M) \simeq \pi_1(M)$ clearly consists of isometries, using the induced Riemannian structure on covering spaces we defined above. Since $\mathbb{H}^n$ is the universal cover, we may write $M = G\backslash\mathbb{H}^n$. □

In particular, if M is a closed hyperbolic manifold, then the action $\pi_1(M) \curvearrowright \mathbb{H}^n$ satisfies the hypotheses of the Milnor-Švarc lemma, so $\pi_1(M)$ is quasiisometric to $\mathbb{H}^n$. We have proven the following.

Proposition 4.9. *The fundamental group $\pi_1(M)$ of a closed hyperbolic n-manifold is (Gromov) hyperbolic.*

4.2. **Ensuring nice actions.** We will be particularly interested in studying the situation where $G = \pi_1(M)$ is such a hyperbolic group containing a codimension-1 subgroup, since this is how we generate an action. For this action to be nice, we will further ask for the subgroup to be *quasiconvex*.

Definition 4.10 (Quasiconvex Subspace)**.** Let (X, d) be a geodesic metric space. A subspace $Y \subseteq X$ is quasiconvex if there is some $k > 0$ such that if p is a geodesic in X connecting two points in Y, then $\mathrm{im}(p) \subseteq N_k(Y)$.

Then, if G is a finitely generated by $S \subseteq G$, we will say a subgroup $H \subseteq G$ is *quasiconvex* if it is so in the Cayley graph $\mathcal{G}(G, S)$. Given a quasiconvex codimension-1 subgroup $H \subseteq G$, where G is hyperbolic, we will define the *limit set* of H to be[4]

$$\Lambda(H) = \overline{Hx} \cap \partial G.$$

Here, Hx is an arbitrary orbit of some point $x \in \mathcal{G}(G, S)$ and ∂G is the *ideal boundary*.

Remark 4.11. If G is $\pi_1(M)$, where M is a closed hyperbolic n-manifold, then G is quasiisometric to $\mathbb{H}^n$, so $\Lambda(H)$ lives in $\mathbb{S}^{n-1} = \partial\mathbb{H}^n$.

We know that from a codimension-1 subgroup $H \subseteq G$, we can build an CAT(0) cube complex $X(\Sigma)$ such that $G \curvearrowright X(\Sigma)$. In [GMRS98], the authors show that quasiconvexity ensures this action is cocompact.

Proposition 4.12. *If G is hyperbolic and $H \subseteq G$ is a quasiconvex codimension-1 subgroup, then the action of G on $X(\Sigma)$ is cocompact.*

In fact, Sageev showed that doing the same construction, but for an arbitrary finite collection of codimension-1 subgroups $H_1, H_2, \ldots, H_k \subseteq G$, gives an analogous result [Sag97]. The following theorem due to Kahn and Markovic will be integral [KM12].

Theorem 4.13. *If M is a closed hyperbolic 3-manifold, then any pair of points $x, y \in \partial\pi_1(M) \simeq \mathbb{S}^2$ is separated by $\Lambda(H)$ for some quasiconvex codimension-1 subgroup $H \subseteq \pi_1(M)$.*[5]

Finally, a theorem of Bergeron and Wise gives properness [BW12].

[4]For more details about $\Lambda(H)$, see [BH99], [Gro87], and [Swe01].

[5]By "separated by," we mean that x and y live in distinct components of $\partial\pi_1(M) \setminus \Lambda(H)$.

Theorem 4.14. *If G is hyperbolic and every pair of points $x, y \in \partial G$ are separated by $\Lambda(H)$, for some quasiconvex codimension-1 $H \subseteq G$, then there are finitely many quasiconvex codimension-1 subgroups $H_1, H_2, \ldots, H_k \subseteq G$ such that the action $G \curvearrowright X(\Sigma)$ is proper.*

We can conclude the following.

Theorem 4.15. *Every closed hyperbolic 3-manifold M acts properly and cocompactly on a* CAT(0) *cube complex.*

Proof. Per Proposition 4.9, we know that $\pi_1(M)$ is hyperbolic, and Theorem 4.13 showed we can separate points in $\partial\pi_1(M)$ by $\Lambda(H)$, for a codimension-1 quasiconvex $H \subseteq G$, so we may apply Theorem 4.14. We get cocompactness from Proposition 4.12. □

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