Stabilizer formalism in ∞-qubit systems

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Overview

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Quantum computation is inherently susceptible to quantum noise–families of potential errors triggered by quantum mechanical phenomena.

So, what is noise?

Knill and Laflamme say: a (quantum) channel!

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Alright, then what is a quantum channel?

Fix a state space $\mathfrak{A}$.

Maybe a qubit $\mathbb{C}^2$.

Or better yet, maybe n qubits

$$\mathbb{C}^2 \underbrace{\otimes \cdots \otimes}_{n} \mathbb{C}^2.$$

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A channel on $\mathfrak{A}$ is a bounded linear map on $\mathcal{S}_1(\mathfrak{A})$ of a specific form.

Start with the bounded operators $\mathfrak{A} \to \mathfrak{A}$ of finite trace.[1]

$$\mathcal{S}_1(\mathfrak{A})$$

[1] This is a Banach space.

Couple with the environment $\mathfrak{B}$.

$$\begin{array}{c} \mathcal{S}_1(\mathfrak{A}) \\ \Big\downarrow {\scriptstyle -\otimes\sigma} \\ \mathcal{S}_1(\mathfrak{A}\otimes\mathfrak{B}) \end{array}$$

Evolve unitarily.

$$\begin{array}{c} \mathcal{S}_1(\mathfrak{A}) \\ \big\downarrow {\scriptstyle -\otimes\sigma} \\ \mathcal{S}_1(\mathfrak{A}\otimes\mathfrak{B}) \\ \big\downarrow {\scriptstyle U(-)U^\dagger} \\ \mathcal{S}_1(\mathfrak{A}\otimes\mathfrak{B}) \end{array}$$

Trace off the environment.

$$\begin{array}{c} \mathscr{S}_1(\mathfrak{A}) \\ \Big\downarrow {\scriptstyle -\otimes\sigma} \\ \mathscr{S}_1(\mathfrak{A}\otimes\mathfrak{B}) \\ \Big\downarrow {\scriptstyle U(-)U^\dagger} \\ \mathscr{S}_1(\mathfrak{A}\otimes\mathfrak{B}) \\ \Big\downarrow {\scriptstyle \mathrm{tr}_{\mathfrak{B}}} \\ \mathscr{S}_1(\mathfrak{A}) \end{array}$$

Every quantum channel $\mathcal{E}$ on $\mathfrak{A}$ decomposes as a sum[1]

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger, \quad E_k \in \mathbb{B}(\mathfrak{A}).$$

We call the E_k the Kraus (or error) operators of $\mathcal{E}$.

[1] If $\dim(\mathfrak{A})$ is infinite, this series converges in trace norm.

Say $\mathscr{C}$ is a (closed) subspace $P : \mathfrak{A} \twoheadrightarrow \mathscr{C} \subseteq \mathfrak{A}$ with orthonormal basis $(|j\rangle)_j$. The Knill-Laflamme correctability condition tells us that a channel $\mathscr{E}$ has a retraction channel $\mathscr{R}$ over $\mathscr{C}$ if and only if

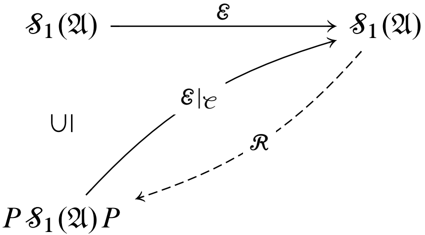
$$\langle j| E_\ell^\dagger E_m |j\rangle = \langle k| E_\ell^\dagger E_m |k\rangle$$

and

$$\langle j| E_\ell^\dagger E_m |k\rangle = 0,$$

for all j, k, ℓ, m.

$$
\begin{array}{ccc}
\mathcal{S}_1(\mathfrak{A}) & \xrightarrow{\quad \mathcal{E} \quad} & \mathcal{S}_1(\mathfrak{A}) \\
\rotatebox{90}{$\subseteq$} & \nearrow_{\mathcal{E}|_{\mathcal{C}}} & \\
P\,\mathcal{S}_1(\mathfrak{A})\,P & &
\end{array}
$$


$\mathcal{S}_1(\mathfrak{A})$
$\mathcal{E}$
$\mathcal{S}_1(\mathfrak{A})$
$\cup|$
$\mathcal{E}|_{\mathcal{C}}$
$\mathcal{R}$
$P\,\mathcal{S}_1(\mathfrak{A})\,P$

The Pauli group $\mathscr{P}$ on 1 qubit is the order-16 matrix group generated by the Pauli spin matrices.

$$\mathscr{P} = \langle i \rangle \{\mathbb{1}, X, Y, Z\} \subseteq \mathrm{GL}_2(\mathbb{C})$$

The *n-qubit Pauli group* $\mathcal{P}_n$ consists of all n-fold (Kronecker) tensor products of elements of $\mathcal{P}$.

$$\mathcal{P}_n = \{\sigma_1 \otimes \cdots \otimes \sigma_n : \sigma_j \in \mathcal{P}\} \subseteq \mathrm{GL}_{2^n}(\mathbb{C})$$

Gottesman's stabilizer formalism lets us do error correction algebraically using the Pauli group.

If $\mathcal{E}$ is an n-qubit channel whose error operators are all from $\mathcal{P}_n$, then $\mathcal{E}$ is correctable on the $\mathcal{S}$-fixed[2] subspace $\mathrm{Fix}_{\mathcal{S}}(\mathfrak{A})$ precisely when

$$E_k^{\dagger} E_{\ell} \in (\mathcal{P}_n \setminus \mathrm{C}_{\mathcal{P}_n}(\mathcal{S})) \cup \mathcal{S}.$$

[2] Here, $\mathcal{S}$ is an abelian subgroup without the negative identity $-\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2$.

Suppose our physical system is a countable collection of coupled qubits. For instance, a chain[3] of spin-1/2 particles.

[3] For convenience, we index the sites by $\mathbb{N} = 1, 2, 3, \ldots$ instead of the more usual choice $\mathbb{Z}$.

Then, our state space should be an infinite tensor product

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots .$$

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Instead, we say an ∞-qubit Hilbert space $\mathfrak{H}$ is spanned by infinite sequences of the (standard) basis of $\mathbb{C}^2$, where we allow finite variation from a chosen basis vector in each copy of $\mathbb{C}^2$.

Define the ∞-qubit Pauli group $\mathcal{P}_\infty$ be the set of tensor sequences of operators in $\mathcal{P}$, where all but finitely many operators are trivial.

$$\mathcal{P}_\infty = \{\sigma_1 \otimes \sigma_2 \otimes \cdots : \sigma_j \in \mathcal{P} \text{ and } \left|\{\sigma_j \neq \mathbb{1}_2\}\right| < \infty\} \subseteq \mathbb{B}(\mathfrak{H}).$$

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Replacing all the ns in the stabilizer formalism with ∞s, we obtain a stabilizer formalism for ∞-qubit systems!

We now ask, “what does $\mathcal{P}_\infty$ look like when we zoom in?”

Surely, if we are good at our jobs, the answer is “like the Pauli group!”

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Observe that there is a sequence of inclusions

$$\mathcal{P} \hookrightarrow \mathcal{P}_2 \hookrightarrow \mathcal{P}_3 \hookrightarrow \cdots \hookrightarrow \mathcal{P}_n \hookrightarrow \cdots$$

where we send

$$\sigma \mapsto \sigma \otimes \mathbb{1}_2$$

in every map.

We are working with groups, so we can take the direct limit

$$\operatorname*{colim}_n(\mathcal{P} \hookrightarrow \mathcal{P}_2 \hookrightarrow \mathcal{P}_3 \hookrightarrow \cdots \hookrightarrow \mathcal{P}_n \hookrightarrow \cdots).$$

This direct limit is exactly (isomorphic to) $\mathscr{P}_\infty$!

Colimits like this come equipped with embeddings

$$\xi_n : \mathcal{P}_n \hookrightarrow \mathcal{P}_\infty,$$

given here by placing the n-qubit system in the nth open slot.

If $n \neq m$, then

$$[\xi_n(\Sigma), \xi_m(\Sigma')] = 0,$$

so $\mathcal{P}_\infty$ exhibits locality.

But, in what sense is $\mathcal{P}_\infty$ a tensor product of infinitely many copies of $\mathcal{P}$?

For that matter, in what sense is $\mathcal{P}_n$ a finite tensor power of $\mathcal{P}$?

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A C_4-group is a group with a compatible action of the cyclic group

$$C_4 = \langle i \rangle = \{\pm 1, \pm i\}.$$

Notably, this includes all our Pauli groups.

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This language lets us define a tensor product operation

$$\otimes : \mathsf{Grp}(C_4) \times \mathsf{Grp}(C_4) \to \mathsf{Grp}(C_4),$$

constructed using a good notion of C_4-bilinearity.

Using this C_4-tensor product, we can say that

$$\mathcal{P}_n \simeq \mathcal{P} \underbrace{\otimes \cdots \otimes}_{n} \mathcal{P}$$

as C_4-groups.

Then, we can extend this to an isomorphism

$$\mathcal{P}_\infty \simeq \mathcal{P}^{\otimes\infty},$$

where $\mathcal{P}^{\otimes\infty}$ is the (finitely supported) infinite C_4-tensor power of $\mathcal{P}$.

So not only does the ∞-qubit Pauli group behave n-locally like the finite Pauli groups, but 1-locally too!

Most things we have done here are rather general.

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Further reading:

(L) Felix Leditzky's notes *Quantum Channels*.

(NC) Nielsen and Chuang's book *Quantum Computation and Quantum Information*, sections §8 and §10.

(KLM) Kaye, Laflamme, and Mosca's book *An Introduction to Quantum Computing*, section §10.

References

Stephane Attal.
Quantum Channels.
Lecture Notes, Institut Camille Jordan, University of Lyon., 2013.

Daniel Gottesman.
Stabilizer Codes and Quantum Error Correction, 1997.

Emanuel Knill and Raymond Laflamme.
Theory of quantum error-correcting codes.
Physical Review A, 55:900–911, Feb 1997.

Karl Kraus.
States, Effects, and Operations.
Springer-Verlag Berlin, 1 edition, 1983.

Thanks! Questions?

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