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Stabilizer Formalism
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Extension to Spin Chains
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Final Remarks
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Stabilizer formalism in ∞-qubit systems

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Overview

1. Errors as Channels

2. Stabilizer Formalism

3. Extension to Spin Chains

4. Locality Two Ways

5. Final Remarks

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Quantum computation is inherently susceptible to quantum noise–families of potential errors triggered by quantum mechanical phenomena.

So, what is noise?

Knill and Laflamme say: a (quantum) channel!

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Alright, then what is a quantum channel?

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Fix a state space $\mathfrak{A}$.

Maybe a qubit $\mathbb{C}^2$.

Or better yet, maybe n qubits

$$\mathbb{C}^2 \underbrace{\otimes \cdots \otimes}_{n} \mathbb{C}^2.$$

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A channel on $\mathfrak{A}$ is a bounded linear map on $\mathcal{S}_1(\mathfrak{A})$ of a specific form.

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Start with the bounded operators $\mathfrak{A} \to \mathfrak{A}$ of finite trace.[1]

$$\mathcal{S}_1(\mathfrak{A})$$

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[1] This is a Banach space.

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Couple with the environment $\mathfrak{B}$.

$$\mathcal{S}_1(\mathfrak{A})$$
$$\downarrow {\scriptstyle -\otimes\sigma}$$
$$\mathcal{S}_1(\mathfrak{A} \otimes \mathfrak{B})$$

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Evolve unitarily.

$$\mathcal{S}_1(\mathfrak{A})$$

$$\downarrow {-\otimes\sigma}$$

$$\mathcal{S}_1(\mathfrak{A}\otimes\mathfrak{B})$$

$$\downarrow {U(-)U^\dagger}$$

$$\mathcal{S}_1(\mathfrak{A}\otimes\mathfrak{B})$$

Trace off the environment.

$$\mathcal{S}_1(\mathfrak{A})$$

$$\Big\downarrow {-\otimes\sigma}$$

$$\mathcal{S}_1(\mathfrak{A}\otimes\mathfrak{B})$$

$$\Big\downarrow {U(-)U^\dagger}$$

$$\mathcal{S}_1(\mathfrak{A}\otimes\mathfrak{B})$$

$$\Big\downarrow {\mathrm{tr}_\mathfrak{B}}$$

$$\mathcal{S}_1(\mathfrak{A})$$

Every quantum channel $\mathcal{E}$ on $\mathfrak{A}$ decomposes as a sum[1]

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger, \quad E_k \in \mathbb{B}(\mathfrak{A}).$$

We call the E_k the Kraus (or error) operators of $\mathcal{E}$.

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[1]If $\dim(\mathfrak{A})$ is infinite, this series converges in trace norm.
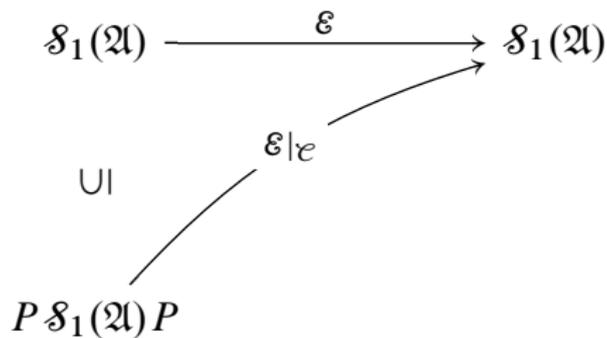
Say $\mathcal{C}$ is a (closed) subspace $P : \mathfrak{A} \twoheadrightarrow \mathcal{C} \subseteq \mathfrak{A}$ with orthonormal basis $(|j\rangle)_j$. The Knill-Laflamme correctability condition tells us that a channel $\mathcal{E}$ has a retraction channel $\mathcal{R}$ over $\mathcal{C}$ if and only if
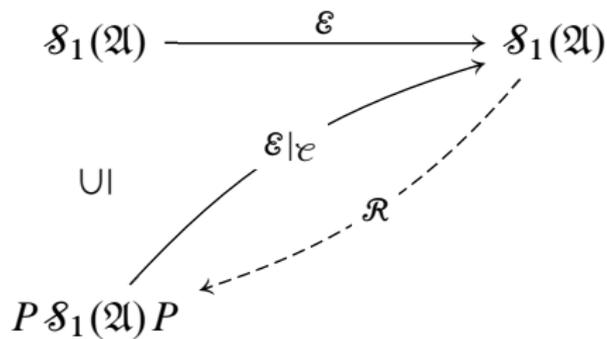
$$\langle j | E_\ell^\dagger E_m | j \rangle = \langle k | E_\ell^\dagger E_m | k \rangle$$

and

$$\langle j | E_\ell^\dagger E_m | k \rangle = 0,$$

for all j, k, ℓ, m.

$$\mathcal{S}_1(\mathfrak{A}) \xrightarrow{\quad\mathcal{E}\quad} \mathcal{S}_1(\mathfrak{A})$$

$$\mathcal{E}|_{\mathcal{C}}$$

$$\cup|$$

$$P\,\mathcal{S}_1(\mathfrak{A})\,P$$

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$\mathcal{E}|_{\mathcal{C}}$

$\cup$|

$\mathcal{R}$

$$P\,\mathcal{S}_1(\mathfrak{A})\,P$$

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The Pauli group $\mathcal{P}$ on 1 qubit is the order-16 matrix group generated by the Pauli spin matrices.

$$\mathcal{P} = \langle i \rangle \{ \mathbb{1}, X, Y, Z \} \subseteq \mathrm{GL}_2(\mathbb{C})$$

The n-qubit Pauli group $\mathcal{P}_n$ consists of all n-fold (Kronecker) tensor products of elements of $\mathcal{P}$.

$$\mathcal{P}_n = \{\sigma_1 \otimes \cdots \otimes \sigma_n : \sigma_j \in \mathcal{P}\} \subseteq \mathrm{GL}_{2^n}(\mathbb{C})$$

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Gottesman's stabilizer formalism lets us do error correction algebraically using the Pauli group.

If $\mathcal{E}$ is an n-qubit channel whose error operators are all from $\mathcal{P}_n$, then $\mathcal{E}$ is correctable on the $\mathcal{S}$-fixed[2] subspace $\mathrm{Fix}_{\mathcal{S}}(\mathfrak{A})$ precisely when

$$E_k^{\dagger} E_{\ell} \in (\mathcal{P}_n \setminus \mathrm{C}_{\mathcal{P}_n}(\mathcal{S})) \cup \mathcal{S}.$$

_____

[2]Here, $\mathcal{S}$ is an abelian subgroup without the negative identity $-\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2$.

Suppose our physical system is a countable collection of coupled qubits.
For instance, a chain[3] of spin-1/2 particles.

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[3]For convenience, we index the sites by $\mathbb{N} = 1, 2, 3, \dots$
instead of the more usual choice $\mathbb{Z}$.

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Then, our state space should be an infinite tensor product

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots .$$

But this isn't a Hilbert space!

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Instead, we say an ∞-qubit Hilbert space $\mathfrak{H}$ is spanned by infinite sequences of the (standard) basis of $\mathbb{C}^2$, where we allow finite variation from a chosen basis vector in each copy of $\mathbb{C}^2$.

Define the ∞-qubit Pauli group $\mathcal{P}_\infty$ be the set of tensor sequences of operators in $\mathcal{P}$, where all but finitely many operators are trivial.

$$\mathcal{P}_\infty = \{\sigma_1 \otimes \sigma_2 \otimes \cdots : \sigma_j \in \mathcal{P} \text{ and } \left|\{\sigma_j \neq \mathbb{1}_2\}\right| < \infty\} \subseteq \mathbb{B}(\mathfrak{H}).$$

Replacing all the ns in the stabilizer formalism with ∞s, we obtain a stabilizer formalism for ∞-qubit systems!

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We now ask, "what does $\mathcal{P}_\infty$ look like when we zoom in?"

Surely, if we are good at our jobs, the answer is "like the Pauli group!"

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Observe that there is a sequence of inclusions

$$\mathcal{P} \hookrightarrow \mathcal{P}_2 \hookrightarrow \mathcal{P}_3 \hookrightarrow \cdots \hookrightarrow \mathcal{P}_n \hookrightarrow \cdots$$

where we send

$$\sigma \mapsto \sigma \otimes \mathbb{1}_2$$

in every map.

We are working with groups, so we can take the red direct limit

$$\operatorname*{colim}_{n}(\mathscr{P} \hookrightarrow \mathscr{P}_2 \hookrightarrow \mathscr{P}_3 \hookrightarrow \cdots \hookrightarrow \mathscr{P}_n \hookrightarrow \cdots).$$

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This direct limit is exactly (isomorphic to) $\mathcal{P}_\infty$!

Colimits like this come equipped with embeddings

$$\xi_n : \mathcal{P}_n \hookrightarrow \mathcal{P}_\infty,$$

given here by placing the n-qubit system in the nth open slot.

If $n \neq m$, then

$$[\xi_n(\Sigma), \xi_m(\Sigma)] = 0,$$

so $\mathcal{P}_\infty$ exhibits locality.

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But, in what sense is $\mathcal{P}_\infty$ a tensor product of infinitely many copies of $\mathcal{P}$?

For that matter, in what sense is $\mathcal{P}_n$ a finite tensor power of $\mathcal{P}$?

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But, in what sense is $\mathcal{P}_\infty$ a tensor product of infinitely many copies of $\mathcal{P}$?

For that matter, in what sense is $\mathcal{P}_n$ a finite tensor power of $\mathcal{P}$?

A C_4-group is a group with a compatible action of the cyclic group

$$C_4 = \langle i \rangle = \{\pm 1, \pm i\}.$$

Notably, this includes all our Pauli groups.

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This language lets us define a tensor product operation

$$\otimes : \mathsf{Grp}(C_4) \times \mathsf{Grp}(C_4) \to \mathsf{Grp}(C_4),$$

constructed using a good notion of C_4-bilinearity.

Using this C_4-tensor product, we can say that

$$\mathcal{P}_n \simeq \mathcal{P} \underbrace{\otimes \cdots \otimes}_{n} \mathcal{P}$$

as C_4-groups.

Then, we can extend this to an isomorphism

$$\mathcal{P}_\infty \simeq \mathcal{P}^{\otimes\infty},$$

where $\mathcal{P}^{\otimes\infty}$ is the (finitely supported) infinite C_4-tensor power of $\mathcal{P}$.

So not only does the ∞-qubit Pauli group behave n-locally like the finite Pauli groups, but 1-locally too!

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Most things we have done here are rather general.

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Further reading:

(L) Felix Leditzky's notes *Quantum Channels*.

(NC) Nielsen and Chuang's book *Quantum Computation and Quantum Information*, sections §8 and §10.

(KLM) Kaye, Laflamme, and Mosca's book *An Introduction to Quantum Computing*, section §10.

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Thanks! Questions?

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