MATH 405: TEACHER'S COURSE, HONORS PROJECT 01
LINEAR SYSTEMS VIA LINEAR SPACES

Summary. When asked what "algebra" is, the first thing that may come to mind is the task of finding roots of multivariable polynomials $p(x_1, \dots, x_n)$ of various degrees. When we restrict to degree $\deg(p) = 1$, we obtain a *linear equation*

$$p(x_1, \dots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + a_{n+1} = 0.$$

When the variables $x_1, \dots, x_n$ are constrained by more than one such linear equation, we have what is called an *imhomogeneous system*, usually writing $p(x_1, \dots, x_n) - a_{n+1} = b$. When $b = 0$, we have a *homogeneous system*.

In this worksheet, you will begin by recalling the two usual methods of solving a linear system from early algebra. For example, these methods have been tested on standardized tests like the SAT. You will gradually repackage some of your solutions in the context of subspaces of $\mathbb{C}^n$. You will then be asked to generalize these ideas slightly to the setting of abstract complex vector spaces. Note that we will assume all notions are $\mathbb{C}$-linear, i.e., all coefficients and constants are complex. You may assume all $\mathbb{C}$-linear vector spaces are finite-dimensional.

Exercise 1 (Substitution)**.** Consider the system

$$\begin{cases} p_1(x_1, x_2) &= 3x_1 + 4ix_2 = 0 \\ p_2(x_1, x_2) &= -\frac{1}{4}ix_1 + x_2 = 2. \end{cases}$$

Manipulate the second equation $p_2(x_1, x_2) = 2$ to obtain a formula for x_2. Then, plug this into the first equation $p_1(x_1, x_2) = 0$ and solve for x_1. Finally, use this to solve for x_2. Write your final answer as a pair (x_1, x_2).

Solution.

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Exercise 2 (Elimination)**.** Consider the system

$$\begin{cases} p_1(x_1, x_2, x_3) & = x_1 + 2ix_2 + 6x_3 = 5 \\ p_2(x_1, x_2, x_3) & = -4ix_2 + 12x_3 = 4 \\ p_3(x_1, x_2, x_3) & = 4x_3 = i. \end{cases}$$

Add $\frac{1}{2}p_2(x_1, x_2, x_3) = \frac{1}{2}(4)$ to $p_1(x_1, x_2, x_3) = 5$ to obtain a new equation $q_1(x_1, x_2, x_3) = 7$. Add $-3p_3(x_1, x_2, x_3) = -3i$ to $p_2(x_1, x_2, x_3) = 4$ to obtain a new equation $q_2(x_1, x_2, x_3) = 4 - 3i$. Solve the easier system

$$\begin{cases} q_1(x_1, x_2, x_3) = 7 \\ q_2(x_1, x_2, x_3) = 4 - 3i \\ p_3(x_1, x_2, x_3) = i \end{cases}$$

using some simple division to obtain a solution (x_1, x_2, x_3) to our original system.

Solution.

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Exercise 3 (Underconstrained)**.** Consider the homogeneous system

$$\begin{cases} p_1(x_1, x_2, x_3) &= 2x_1 + 8ix_2 = 0 \\ p_2(x_1, x_2, x_3) &= 4ix_2 = 0. \end{cases}$$

How many solutions to the above system are there? Specifically, write down the subset of triples $(x_1, x_2, x_3) \in \mathbb{C}^3$ that give solutions. Let m be the number of equations in our system and n the number of variables/unknowns. Write an inequality between m and n that seems to explain this phenomenon.

Solution.

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Exercise 4 (Matrix Form)**.** Given a system

$$\begin{cases} p_1(x_1,\dots,x_n) & = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ p_2(x_1,\dots,x_n) & = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ & \vdots \\ p_m(x_1,\dots,x_n) & = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

with n unknowns and m equations, we can organize the data of the system into a matrix equation

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{b}.$$

In particular, when the system is homogeneous, we can write this as $Ax = 0_m$, so the data of a homogeneous system is a matrix $A \in \mathbb{M}_{m\times n}(\mathbb{C})$. First, do the matrix multiplication to confirm the matrix equation recovers the system. Then, write the system from Exercise 2 as a matrix equation $Ax = b$. Explain how to solve this system in matrix form by doing an analogue of elimination on the *augmented matrix*

$$(A \mid b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

Solution.

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Exercise 5. The quantity $\operatorname{rank}(A)$ of a matrix A is the number of linearly independent columns (or rows). Every $m \times n$ matrix $A \in \mathbb{M}_{m\times n}(\mathbb{C})$ can equivalently be viewed as a linear map

$$A : \mathbb{C}^n \to \mathbb{C}^m.$$

Then, it is a quick check to show that $\operatorname{rank}(A)$ is the dimension $\dim(\operatorname{image}(A))$, where

$$\operatorname{image}(A) = \{w \in \mathbb{C}^m : Av = w \text{ for some } v \in \mathbb{C}^n\} \subseteq \mathbb{C}^m.$$

Since $\operatorname{image}(A) \subseteq \mathbb{C}^m$, we know that

$$\operatorname{rank}(A) = \dim(\operatorname{image}(A)) \leq m.$$

Generally, a homogeneous system given by A is said to be *rank deficient* when $\operatorname{rank}(A) < n$. Show that when there are more unknowns than equations in a homogeneous system A, as we saw in Exercise 3, the system is rank deficient. (*Hint: This amounts to combining two inequalities.*)

Solution.

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Exercise 6 (Kernels)**.** Given a linear map $T : \mathcal{V} \to \mathcal{W}$, its kernel is the subspace

$$\ker(T) = \{v \in \mathcal{V} : Tv = 0_{\mathcal{W}}\} \subseteq \mathcal{V}.$$

Compute the kernel $\ker(A)$, where $A : \mathbb{C}^3 \to \mathbb{C}^2$ is the matrix corresponding to the system in Exercise 3. (*Hint: This system is homogeneous.*)

Solution.

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Exercise 7 (Rank-Nullity)**.** The quantity nullity(T) of a linear transformation is the dimension

$$\text{nullity}(T) = \dim(\ker(T)).$$

The *rank-nullity theorem* is the statement that there is an isomorphism

$$\mathcal{V} \simeq \text{image}(T) \oplus \ker(T),$$

which gives the relation

$$\dim(\mathcal{V}) = \text{rank}(T) + \text{nullity}(T).$$

Using this relation, compare the dimensions of the vector space of solutions to a rank deficient homogeneous system A and a homogeneous system B with $\text{rank}(B) = n$. (*Hint: Write the solution space as a kernel.*) Confirm that this agrees with your solution to the previous exercise.

Solution.

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Exercise 8. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear map between finite-dimensional vector spaces.

(i) Explain how to write T as a matrix $[T] : \mathbb{C}^{\dim(\mathcal{V})} \to \mathbb{C}^{\dim(\mathcal{W})}$. (*Hint: Pick bases for* $\mathcal{V}$ *and* $\mathcal{W}$.)
(ii) Using matrix form, explain how to compute the kernel $\ker([T])$.
(iii) Does this also compute $\ker(T)$? (*Hint: Is picking a basis linear? Is it reversible?*)

Solution.

(i)

(ii)

(iii)

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