MATH 231: CALCULUS II, HONORS PROJECT 04
EULER'S SOLUTION

Summary. It is finally time to study the first ever solution to the Basel problem, as presented by Euler in 1734. To do so, we will make use of the *Weierstrass factorization* of sine. In this worksheet, you will compare two infinite expressions of the sine function, one additive and one multiplicative. By making this comparison, a solution to the Basel problem will appear.

Exercise 1 (Sine as a Series)**.** Write down the Maclaurin series of $\sin(x)$. Then, show that it has radius of convergence $R = \infty$ using the ratio test.

Solution.

□

Remark 1 (What is. . . an entire function?). Let $\mathbb{C}$ be the collection of all complex numbers $z = a + bi$. We say that a function $f : \mathbb{C} \to \mathbb{C}$ is *entire* if $f(z)$ has a power series expansion that converges everywhere. Now, let the *complex sine* function $\sin : \mathbb{C} \to \mathbb{C}$ just be given by letting $\sin(z)$ equal to the Maclaurin series above (the only difference being your variable is now z, a complex number). Since we showed this series has radius $R = \infty$, we have shown that $\sin(z)$ is an entire function!

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Exercise 2 (Roots of Sine)**.** By drawing a picture, show that the roots of $\sin(x)$ are

$$0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi, \dots, \pm n\pi, \dots.$$

Solution.

□

Remark 2 (Infinite Products). In the same way that we defined the infinite sum/series

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

of a sequence (a_n) as a limit of partial sums

$$\lim_{N\to\infty}(a_1 + a_2 + \cdots + a_N) = \lim_{N\to\infty}\sum_{n=1}^{N} a_n,$$

we define the *infinite product*

$$a_1 a_2 a_3 \cdots = \prod_{n=1}^{\infty} a_n$$

as a limit of *partial products*

$$\lim_{N\to\infty}(a_1 a_2 \cdots a_N) = \lim_{N\to\infty}\prod_{n=1}^{N} a_n.$$

We will now combine Remark 1 and Exercise 2 to use a special case of the Weierstrass factorization theorem.

Theorem 0.1 (Weierstrass Factorization of Sine)**.** *Since* $\sin(z)$ *is entire and has a root at* $z = 0$*, it admits a factorization in terms of its roots* $n\pi$ *for* $n = 1, 2, \ldots$*. Specifically, this factorization can be computed to be*

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right).$$

Exercise 3 (Sine as a Product)**.** Write out the first five factors of

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right).$$

Solution.

□

Exercise 4 (A Term, Two Ways)**.**

(a) Multiply out the first five factors you found above to show that the z^2 terms in the infinite product look like

$$-\frac{z^2}{\pi^2} - \frac{z^2}{4\pi^2} - \frac{z^2}{9\pi^2} - \frac{z^2}{16\pi^2} - \frac{z^2}{25\pi^2} - \cdots.$$

Use this to conclude that generally, the z^2 term in the infinite product is given by

$$-\frac{z^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(b) Divide your Maclaurin series (from Exercise 1) for $\sin(z)$ by z to find a power series expansion of $\sin(z)/z$. Show that the z^2 term in the power series is

$$-\frac{z^2}{6}.$$

(a) *Solution.*

□

(b) *Solution.*

□

Take a look at part (a) of the previous exercise. Once again, the Basel series has appeared! Using our usual trick of "comparing two expressions we know are equal," we can now solve the Basel problem.

Exercise 5 (Solving Basel)**.** We have found two ways of writing

$$\frac{\sin(z)}{z},$$

one as an infinite sum and one as an infinite product. These expressions should agree, so in particular, they should have the same z^2 term. So, set the two z^2 terms you found in the previous exercise equal. Then, rearrange and solve for the Basel series!

Solution.

□

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