MATH 231: CALCULUS II, HONORS PROJECT 03
FOURIER AND PARSEVAL

Summary. Named after French mathematician (and physicist) Joseph Fourier, the *Fourier series* associated to a function f is a (doubly infinite) series with complex coefficients. Parseval's identity gives us a way to relate the (sum of the squares of the) coefficients to the (integral of the square of the) function. In this worksheet, you will be guided through solving the Basel problem using Fourier coefficients, utilizing Parseval's identity along the way. This is a well-known solution, and so there are many variations on it. The specific presentation in this worksheet most closely resembles the blog post solution by Stephen Tu.

Let us set the stage a bit. To use Fourier series, we need a function f. So, let

$$f(t) = \frac{1}{2} - t,$$

where $0 \leq t \leq 1$. That is, we have a function $f : [0, 1] \to \mathbb{R}$.

Exercise 1 (Finite $\mathcal{L}^2$-Norm)**.** Let $\|f\|_{\mathcal{L}^2}$ be the (square rooted) definite integral

$$\|f\|_{\mathcal{L}^2} = \sqrt{\int_0^1 (f(t))^2 \, \mathrm{d}t}.$$

We call $\|f\|_{\mathcal{L}^2}$ the $\mathcal{L}^2$-*norm* of f. Show that the squared integral

$$\|f\|_{\mathcal{L}^2}^2 = \int_0^1 (f(t))^2 \, \mathrm{d}t = \int_0^1 \left(\frac{1}{2} - t\right)^2 \mathrm{d}t$$

is $1/12$.

Solution.

□

Remark 1 (What is... $\mathcal{L}^2$-Space?). Observe that $\|f\|_{\mathcal{L}^2}^2 = 1/12$ is a finite number, so its square root $\|f\|_{\mathcal{L}^2} = 1/\sqrt{12}$ is too. That is, f has finite $\mathcal{L}^2$-norm. The collection/space of all functions $[0, 1] \to \mathbb{R}$ that have this property of "square integrability" is denoted $\mathcal{L}^2([0, 1])$ and is called $\mathcal{L}^2$-*space*.

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Exercise 2 (Euler's Formula)**.** Recall that a complex number is a number of the form $a + bi$, where $i = \sqrt{-1}$ is the imaginary unit and a and b are real numbers. Sometimes, we will want to work with exponential expressions of the form $e^{i\theta}$, where θ is some angle in radians. To compute this number, we tend to use *Euler's formula*

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Show that $e^{-2\pi in} = 1$ for any integer number $n = \ldots, -2, -1, 0, 1, 2, \ldots$ using Euler's formula. This is made easy by remembering that as an angle, -2π is the same as 2π and going all the way around a circle n times gives the same angle as going around 0 times, i.e., plug in $\theta = -2\pi n = 0$, as an angle.

Solution.

□

Exercise 3 (Fourier Coefficients)**.** The nth Fourier coefficient of f is the integral

$$\widehat{f}(n) = \int_0^1 f(t)e^{-2\pi int}\,\mathrm{d}t\,,$$

where i is the imaginary unit and n is an integer, as in the previous exercise.

(a) Plug in $n = 0$ to show that the 0th Fourier coefficient is 0:

$$\widehat{f}(0) = \int_0^1 f(t)e^{-2\pi i(0)t}\,\mathrm{d}t = \int_0^1 f(t)e^0\,\mathrm{d}t = \int_0^1 \left(\frac{1}{2} - t\right)\mathrm{d}t = 0.$$

(b) Use integration by parts to show that for any integer $n \neq 0$, the nth Fourier coefficient is

$$\widehat{f}(n) = \int_0^1 f(t)e^{-2\pi int}\,\mathrm{d}t = \frac{1}{2\pi in}.$$

Hint: For your integration by parts, use $u = (1/2 - t)$, so that $\mathrm{d}u = -1\,\mathrm{d}t$, and $\mathrm{d}v = e^{-2\pi int}\,\mathrm{d}t$, so that $v = 1/(-2\pi in)e^{-2\pi int}$. After using the fundamental theorem of calculus to evaluate the definite integral, you will end up with an expression in terms of $e^{-2\pi in(1)} = e^{-2\pi in}$ and $e^{-2\pi in(0)} = e^0$. Yet, by the previous exercise, $e^{-2\pi in} = 1$, and $e^0 = 1$, so things simplify a lot.

(a) *Solution.*

□

(b) *Solution.*

□

Exercise 4 (Doubly Infinite Series)**.** Recall that when we learned about improper integrals, we said that

$$\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = \int_{-\infty}^{0} g(x)\,\mathrm{d}x + \int_{0}^{\infty} g(x)\,\mathrm{d}x\,,$$

assuming both of the individual improper integrals, which were defined as limits, converge. We did this to separate the "positive part" and "negative part" of $g(x)$. Such an integral could be called *doubly infinite*, since both the upper and lower bound are infinite.

We can do something similar for series! Define the *doubly infinite series* of a sequence a_n to be

$$\sum_{-\infty}^{\infty} a_n = \sum_{n=0}^{\infty} a_n + \sum_{n=1}^{\infty} a_{-n},$$

assuming both of these individual series converge. The first series in the sum is just the usual series from $n = 0$ to ∞ of the sequence a_n. The second series in the sum is the series from $n = 1$ to ∞ of the sequence a_{-n}, i.e., a_n but where we make n negative. Note that we have to start the second series in the sum from $n = 1$, so that we do not double count $a_0 = a_{-0}$ as we add things up.

(a) The *modulus square* of a complex number $a + bi$ is defined as

$$|a + bi|^2 = (a + bi)(a - bi).$$

That is, we take our complex number, and multiply it by the complex number given by replacing i with $(-i)$. Since $-i^2 = -(-1) = 1$, the modulus square is always a (nonnegative) real number. Show that for integers $n \neq 0$,

$$\left|\widehat{f}(n)\right|^2 = \left|\frac{1}{2\pi i n}\right|^2 = \frac{1}{4\pi^2 n^2}$$

(b) Using (a), show that $|\widehat{f}(-n)|^2$ is the same number as $|\widehat{f}(n)|^2$.

(c) Observe that $|\widehat{f}(n)|^2$ gives us a sequence of numbers. We can then ask, what is the doubly infinite series

$$\sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 ?$$

Use the definition of the doubly infinite series above to compute

$$\sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hint: Our doubly infinite series is

$$\sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2 = \sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^2 + \sum_{n=1}^{\infty} \left|\widehat{f}(-n)\right|^2.$$

We can separate the $n = 0$ term out:

$$\left|\widehat{f}(0)\right|^2 + \sum_{n=1}^{\infty} \left|\widehat{f}(n)\right|^2 + \sum_{n=1}^{\infty} \left|\widehat{f}(-n)\right|^2.$$

Now use that in (b), we saw that $|\widehat{f}(n)|^2 = |\widehat{f}(-n)|^2$, so we can simplify the sum of the two series as 2 times one of them:

$$\sum_{n=1}^{\infty} \left|\widehat{f}(n)\right|^2 + \sum_{n=1}^{\infty} \left|\widehat{f}(-n)\right|^2 = 2 \sum_{n=1}^{\infty} \left|\widehat{f}(n)\right|^2,$$

and that in the previous exercise, we learned that the 0th Fourier coefficient is 0.

(a) *Solution.*

□

(b) *Solution.*

□

(c) *Solution.*

□

Take a look at this solution. . . , the Basel series has appeared! We now just need to relate this doubly infinite series of the modulus squared Fourier coefficients to something else so we can solve for the Basel series. This is where Parseval's identity comes into play.

Theorem 0.1 (Parseval's Identity)**.** *The square of the $\mathcal{L}^2$-norm of a function is the same as the doubly infinite series of its modulus squared Fourier coefficients. That is,*

$$\|f\|_{\mathcal{L}^2}^2 = \sum_{-\infty}^{\infty} \left|\widehat{f}(n)\right|^2.$$

Exercise 5 (The Solution)**.** We know the value of $\|f\|_{\mathcal{L}^2}^2$ from the first exercise. We also have an expression for the doubly infinite series of Fourier coefficients from part (c) of the previous exercise. Plug these values into Parseval's identity and solve for

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

the Basel series.

Solution.

□

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