

MATH 231: CALCULUS II, HONORS PROJECT 02
DIFFERENTIATION UNDER THE INTEGRAL SIGN

Summary. In 2023, F.L. Freitas showed that the Basel problem can be solved using a technique called *differentiation under the integral sign*. This technique is often associated with physicist Richard Feynman, though it dates back to Leibniz. In this worksheet, you will be guided through solving the Basel problem, following Freitas' solution.

We begin by looking at the improper integral

$$\int_0^{\infty} \ln(1 + \alpha e^{-x} + e^{-2x}) dx,$$

where $-2 \leq \alpha \leq 2$ is some number.

Exercise 1 (Computing the Derivative). Show that

$$\frac{d}{d\alpha} \ln(1 + \alpha e^{-x} + e^{-2x}) = \frac{e^{-x}}{1 + \alpha e^{-x} + e^{-2x}}$$

by using the chain rule. Note that we are differentiating with respect to α , not x , so e^{-x} and e^{-2x} are just constants.

Solution.

□

Remark 1 (Continuity). Observe that the function

$$\ln(1 + \alpha e^{-x} + e^{-2x})$$

is continuous both when viewed as a function of α and as a function of x . Similarly, the derivative you computed in the previous question

$$\frac{e^{-x}}{1 + \alpha e^{-x} + e^{-2x}}$$

is also continuous both when viewed as a function of α and as a function of x . Remember, we are considering the domain $x > 0$ and really only care about when $-2 \leq \alpha \leq 2$. See the graphs of these functions on Desmos.

The above remark allows us to use differentiation under the integral sign. You can see the Wikipedia page on the general result, but below is it stated in our special case.

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Theorem 0.1 (Feynman's Trick). *Because both the integrand and its derivative with respect to α are continuous in α and x , we can move the derivative inside:*

$$\frac{d}{d\alpha} \int_0^{\infty} \ln(1 + \alpha e^{-x} + e^{-2x}) dx = \int_0^{\infty} \frac{d}{d\alpha} \ln(1 + \alpha e^{-x} + e^{-2x}) dx = \int_0^{\infty} \frac{e^{-x}}{1 + \alpha e^{-x} + e^{-2x}} dx.$$

Exercise 2 (*u*-Substitution). Let $I(\alpha)$ be the function

$$I(\alpha) = \int_0^{\infty} \ln(1 + \alpha e^{-x} + e^{-2x}) dx.$$

The above theorem told us that

$$I'(\alpha) = \int_0^{\infty} \frac{e^{-x}}{1 + \alpha e^{-x} + e^{-2x}} dx.$$

Use *u*-substitution with $u = e^{-x}$ to simplify the integral into the form

$$I'(\alpha) = \int_0^1 \frac{1}{1 + \alpha u + u^2} du.$$

Solution.

□

Exercise 3 (Completing the Square). We can complete the square to write

$$1 + \alpha u + u^2 = \left(u + \frac{\alpha}{2}\right)^2 + \left(1 - \frac{\alpha^2}{4}\right).$$

Since we care about α between -2 and 2 , we know $1 - \alpha^2/4$ is a positive number a^2 . Our integral becomes

$$I'(\alpha) = \int_0^1 \frac{1}{\left(u + \frac{\alpha}{2}\right)^2 + a^2} du.$$

Make a substitution $v = u + \alpha/2$ (the bounds become $\alpha/2$ and $1 + \alpha/2$) and use the formula

$$\int \frac{1}{v^2 + a^2} dv = \frac{1}{a} \arctan\left(\frac{v}{a}\right) + C$$

to evaluate the (definite) integral $I'(\alpha)$. After applying the fundamental theorem of calculus, you should have found

$$I'(\alpha) = \frac{2}{\sqrt{4 - \alpha^2}} \left(\arctan\left(\frac{\alpha + 2}{\sqrt{4 - \alpha^2}}\right) - \arctan\left(\frac{\alpha}{\sqrt{4 - \alpha^2}}\right) \right).$$

Solution.

□

Remark 2 (Using Some Identities). You may (or may not) remember that there is an identity

$$\arctan(x) - \arctan(y) = \arctan\left(\frac{x - y}{1 + xy}\right).$$

Let

$$x = \frac{\alpha + 2}{\sqrt{4 - \alpha^2}} \quad \text{and} \quad y = \frac{\alpha}{\sqrt{4 - \alpha^2}}.$$

Then, our solution from the previous problem becomes

$$I'(\alpha) = \frac{2}{\sqrt{4 - \alpha^2}} \arctan\left(\frac{x - y}{1 + xy}\right).$$

Now, we can compute

$$\begin{aligned} \frac{x-y}{1+xy} &= \frac{\frac{\alpha+2}{\sqrt{4-\alpha^2}} - \frac{\alpha}{\sqrt{4-\alpha^2}}}{1 + \frac{(\alpha+2)\alpha}{4-\alpha^2}} \\ &= \frac{\frac{2}{\sqrt{4-\alpha^2}}}{\frac{4-\alpha^2+\alpha^2+2\alpha}{4-\alpha^2}} \\ &= \frac{2(4-\alpha^2)}{\sqrt{4-\alpha^2}(2(2+\alpha))} \\ &= \frac{(2-\alpha)(2+\alpha)}{(2+\alpha)\sqrt{(2-\alpha)(2+\alpha)}} \\ &= \frac{2-\alpha}{\sqrt{2-\alpha}\sqrt{2+\alpha}} \\ &= \sqrt{\frac{2-\alpha}{2+\alpha}}, \end{aligned}$$

where throughout we use that $4 - \alpha^2 = (2 - \alpha)(2 + \alpha)$. Thus, our derivative looks like

$$I'(\alpha) = \frac{2}{\sqrt{4-\alpha^2}} \arctan\left(\sqrt{\frac{2-\alpha}{2+\alpha}}\right).$$

Exercise 4 (Trigonometric Substitution). We wanted to compute $I(\alpha)$, and we have a formula for its derivative. So, we want to compute the antiderivative

$$I(\alpha) = \int \frac{2}{\sqrt{4-\alpha^2}} \arctan\left(\sqrt{\frac{2-\alpha}{2+\alpha}}\right) d\alpha.$$

Compute this integral by using a trigonometric substitution $\alpha = 2 \cos(\theta)$. Note that $d\alpha = -2 \sin(\theta) d\theta$.

Hint: The tricky part is making the arctan nicer. Well, if $\alpha = 2 \cos(\theta)$, then

$$\sqrt{\frac{2-\alpha}{2+\alpha}} = \sqrt{\frac{2-2\cos(\theta)}{2+2\cos(\theta)}} = \sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}.$$

Then, you can use the identities

$$1 - \cos(\theta) = 2 \sin^2(\theta/2) \quad \text{and} \quad 1 + \cos(\theta) = 2 \cos^2(\theta/2)$$

to simplify

$$\arctan\left(\sqrt{\frac{2-\alpha}{2+\alpha}}\right) = \arctan\left(\frac{\sin(\theta/2)}{\cos(\theta/2)}\right) = \arctan(\tan(\theta/2)) = \frac{\theta}{2}.$$

After doing the trigonometric substitution and substituting back in $\theta = \arccos(\alpha/2)$, you should get

$$I(\alpha) = -\frac{1}{2} \arccos\left(\frac{\alpha}{2}\right) + C.$$

Solution.

□

Remark 3 (Factoring the Integrand). We now want to find the constant of integration C . Consider $I(2)$. By the original formula for $I(\alpha)$, this is equal to

$$I(2) = \int_0^\infty \ln(1 + 2e^{-x} + e^{-2x}) \, dx.$$

Well, we can factor the integrand into $1 + 2e^{-x}e^{-2x} = (1 + e^{-x})^2$:

$$I(2) = \int_0^\infty \ln((1 + e^{-x})^2) \, dx = 2 \int_0^\infty \ln(1 + e^{-x}) \, dx,$$

using the log identity $\ln(b^p) = p \ln(b)$.

Exercise 5 (u -Substitution, Again). Do a u -substitution $u = x/2$ to the previous expression for $I(2)$ to get that

$$I(2) = 4 \int_0^\infty \ln(1 + e^{-2u}) \, du = 4I(0).$$

Solution.

□

Exercise 6 (Computing C). Using the formula we found

$$I(\alpha) = -\frac{1}{2} \arccos\left(\frac{\alpha}{2}\right)^2 + C,$$

plugging in $\alpha = 2$ gives us

$$I(2) = 0 + C = C.$$

Similarly, we can plug in $\alpha = 0$ to get

$$I(0) = -\frac{1}{2} \arccos(0)^2 + C.$$

By the previous exercise, we know that

$$I(2) = 4I(0),$$

so

$$C = 4 \left(-\frac{1}{2} \arccos(0)^2 + C \right).$$

Solve this equation to show that our constant of integration is

$$C = \frac{\pi^2}{6}.$$

Solution.

□

Exercise 7 ($I(-2)$, Two Ways).

(a) Use the formula

$$I(\alpha) = -\frac{1}{2} \arccos\left(\frac{\alpha}{2}\right)^2 + C$$

to show that

$$I(-2) = -\frac{\pi^2}{3}.$$

(b) Now, use our original formula

$$I(\alpha) = \int_0^{\infty} \ln(1 + \alpha e^{-x} + e^{-2x}) dx$$

to show that

$$I(-2) = 2 \int_0^{\infty} \ln(1 - e^{-x}) dx.$$

Hint: Like when we computed $I(2)$, we can factor the integrand by writing

$$\ln(1 - 2e^{-x} + e^{-2x}) = \ln((1 - e^{-x})^2).$$

(c) Using parts (a) and (b), conclude that

$$2 \int_0^{\infty} \ln(1 - e^{-x}) dx = -\frac{\pi^2}{3}.$$

Rearranging tells us that

$$-\int_0^{\infty} \ln(1 - e^{-x}) dx = \frac{\pi^2}{6}.$$

(a) *Solution.*

□

(b) *Solution.*

(c) *Solution.*

□

□

Remark 4 (The Key Step). [Note: This remark requires material from LECTURE 20]

Until this point, we have just been looking at this seemingly random integral $I(\alpha)$. We computed it by differentiating under the integral sign, found the constant of integration, and computed $I(-2)$ in two different ways to show that

$$-\int_0^{\infty} \ln(1 - e^{-x}) \, dx = \frac{\pi^2}{6}.$$

So what? The trick is to think of power series. Based on our knowledge of geometric series, we can write

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Integrating the left-hand side gives us

$$-\ln(1-x)$$

and integrating the right-hand side gives

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Now, plug in e^{-x} for x in both expressions and set them equal:

$$-\ln(1 - e^{-x}) = \sum_{n=1}^{\infty} \frac{(e^{-x})^n}{n}.$$

Using our expression with the integral above, we have that

$$-\int_0^{\infty} \ln(1 - e^{-x}) \, dx = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(e^{-x})^n}{n} \, dx = \frac{\pi^2}{6}$$

Exercise 8 (Back to Basel). Show that

$$\int_0^{\infty} \frac{(e^{-x})^n}{n} \, dx = \frac{1}{n^2}.$$

Hint: Write $(e^{-x})^n = e^{-nx}$, use a u -substitution with $u = -nx$ and $du = -n \, dx$, and then use the fundamental theorem of calculus (taking a limit for the upper bound, since this is an improper integral).

Solution.

□

Exercise 9 (The Solution). Combine the conclusions of the previous remark and exercise to solve the Basel problem, i.e., to find the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution.

□