

MATH 231: CALCULUS II, HONORS PROJECT STATES, EVOLUTIONS, AND SOME HISTORY

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ABSTRACT. We give a concise introduction to quantum theory, beginning with history and heuristics, after which we single out the data of a quantum theory and introduce the major players of this project. We also take a moment to review complex arithmetic and look at contemporary approaches.

The observations of Planck (1900), Einstein (1905), and Bohr (1913) led Werner Heisenberg (1925) and Erwin Schrödinger (1926) to develop their respective foundations of quantum theory. Using contemporary language, Heisenberg's approach amounted to a theory of unitary *time evolution of observables*

$$X \xrightarrow{U_t} Y,$$

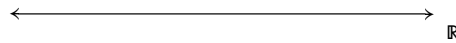
whereas Schrödinger's theory described the *dynamics* of a state ψ based on a *Hamiltonian* or energy observable H . Heisenberg's framework was developed with his mentor Max Born, who subsequently (1926) published a probabilistic interpretation of Schrödinger's theory. Namely, the states of a quantum system were to be given by probability distributions spread over the possible measurement values of an observable.¹ Later, John von Neumann (1932) formalized quantum mechanics via a powerful theory of operators on *Hilbert spaces*, named for David Hilbert, setting a precedent for the language of quantum theory for years to come. In particular, von Neumann's observations and a powerful result due to Marshall Stone (1930, 1932) showed a precise correspondence

$$U_t = e^{-itH}.$$

between Heisenberg's time evolution operators U_t and Schrödinger's Hamiltonian H .²

1. ASIDE ON COMPLEX NUMBERS

Before we start doing quantum mechanics, let us take a moment to familiarize ourselves with some complex arithmetic. Recall that a real number is a point on the *real line*:



For example, any time $t \in \mathbb{R}$ is a point on a timeline, and any location in three dimensions $(x, y, z) \in \mathbb{R}^3$ takes points x , y , and z from three position lines.

To make the mathematics of quantum things cleaner, we tend to utilize *complex numbers* as well. A complex number z is a number of the form

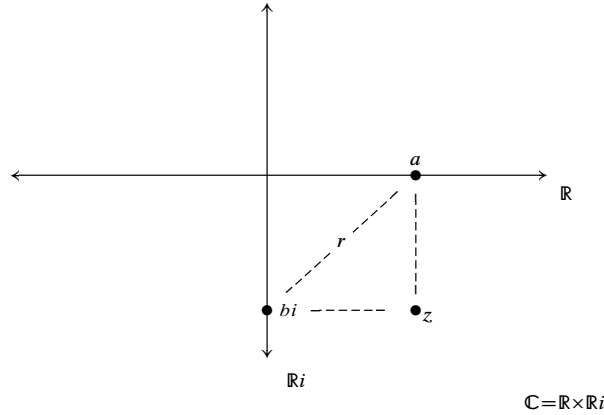
$$z = a + bi,$$

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¹A probability distribution over a space of mutually exclusive events Ω is a collection of nonnegative real numbers p_ω indexed by Ω so that $\int_\Omega p_\omega = 1$.

²The assignment $t \mapsto U_t$ is called a *one-parameter subgroup*.

where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ is the formal square root of -1 .³ We call a the *real part* of z and b the *imaginary part*. Since each a and b are points on a real line, the space of complex numbers $\mathbb{C}$ should look like a plane $\mathbb{R}^2$. We call this space the *complex plane*:



You may have noticed that we have formed a right triangle in the complex plane, and that the hypotenuse r is the same as the distance of z from the origin. By the Pythagorean theorem, this means $a^2 + b^2 = r^2$, and we call $r = |z|$ the *modulus* of z .

Note that to any complex number $z = a + bi$, we can associate another complex number $z^* = a - bi$, called its *conjugate*, by flipping over the real axis in the complex plane. Then, we can compute

$$\begin{aligned} zz^* &= (a + bi)(a - bi) \\ &= a^2 - bi + bi - b^2 i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2 \\ &= |z|^2. \end{aligned}$$

So, the modulus squared of a complex number is exactly itself multiplied by its conjugate.

2. THE DATA OF A QUANTUM THEORY

So, what makes a quantum theory? From our brief glance at the history, a quantum theory of a particle in space should have

- (i) a space of states. Given a quantum system at a time $t \in \mathbb{R}$, its state is described by a wavefunction ψ that assigns to each position x , a complex number $\psi(x, t)$. In other words, the states live in

$$\mathcal{L} = \{\psi(-, t) : \mathbb{R}^n \rightarrow \mathbb{C}\}.$$

To transition from time t to time t' , we apply one of Heisenberg's unitaries.

- (ii) for each state $\psi \in \mathcal{L}$ and time t , a probability

$$\text{Prob}(\psi(x, t)) = |\psi(x, t)|^2,$$

where $|\psi(x, t)|$ is the modulus of the complex number $\psi(x, t)$. We interpret this as the probability of the particle being in the position x .

³We call i the *imaginary unit*.

For these numbers to actually form a probability distribution, we need them to add up to 1:

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1.$$

This integral is often denoted $\|\psi\|_2^2$, and is called the squared $\mathcal{L}^2$ -norm of the wavefunction.

We thus tend to work with $\mathcal{L}^2 \subset \mathcal{L}$, the space of all wavefunctions for which the above improper integral $\|\psi\|_2^2$ converges to a finite value. Note that $\psi(x, t)$ forms a probability distribution over the *position observable*. We could have equivalently formed a wavefunction $\varphi(p, t)$, where p is the momentum of the particle, giving us a probability distribution over the *probability observable*.⁴

- (iii) dynamics of our states dictated by a Hamiltonian H . We want to say that ψ evolves in time proportionally to some function H that describes energy:

$$\frac{\partial}{\partial t} \psi \propto H\psi,$$

where $\partial/\partial t$ is the derivative of ψ with respect to t , holding position x fixed. We call this the *partial derivative* of $\psi(x, t)$ with respect to t . This proportionality can be made exact:

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi.$$

Here, i is the imaginary unit and $\hbar$ is a number called *Planck's constant*, which carries units of energy seconds.

3. TIME INDEPENDENCE

The equation in (iii) is called the *Schrödinger equation*. Sometimes the adjective *time-dependent* is tacked on for the following reason. First, restrict attention to a particle of mass m moving in one dimension of space, i.e., our space of wavefunctions is

$$\mathcal{L}^2 = \{\psi : \mathbb{R} \rightarrow \mathbb{C} : \|\psi\|_2^2 < \infty\}.$$

We assume that our particle is subject to potential energy $V(x)$, and in this case, our Hamiltonian takes on the special form

$$H = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x)\psi(x, t).$$

Substituting this into the time-dependent Schrödinger equation, we find

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) + V(x)\psi(x, t).$$

We have partial derivatives with respect to both space and time here, which is a bit of a headache. However, we are often able to assume that

$$\psi(x, t) = e^{-i\omega t} \psi(x),$$

⁴To move between these two pictures, we would use a unitary called the *Fourier transform*.

separating off the time-dependent part.⁵ Then, we can rewrite the Schrödinger equation as

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} e^{-i\omega t} \psi(x) &= H e^{-i\omega t} \psi(x) \\ i\hbar(-i\omega) e^{-i\omega t} \psi(x) &= H e^{-i\omega t} \psi(x) \\ \hbar\omega \psi(x) &= H \psi(x). \end{aligned}$$

We abbreviate $E = \hbar\omega$, where E stands for energy, since E has units of energy $\times$ time $\times$ 1/time. Our goal is thus to solve

$$H \psi(x) = E \psi(x)$$

for $\psi(x)$.

4. VISTAS

There are many beautiful (and messy) directions we could take, but to be able to use our tools from Calculus II, we will focus on solving the Schrödinger equation for the dynamics of a particle in one-dimensional space. Here are some other directions you may find interesting:

- (i) The abstract theory of spaces and functions on them is, in part, a generalization of the Calculus II tools you know to an infinite-dimensional setting. Given two wavefunctions $\psi, \varphi \in \mathcal{L}^2$, we can always compute the quantity

$$\langle \psi | \varphi \rangle = \int_{-\infty}^{\infty} \psi(x, t)^* \varphi(x, t) dx,$$

called the *inner product*. An active area of research, *operator algebra*, studies the functions or operators on spaces equipped with an inner product, called Hilbert spaces.

- (ii) The data of a quantum theory involved assigning to each quantum system a Hilbert space, and to each transition in time $t \rightarrow t'$, a corresponding unitary operator on that Hilbert space. Studying these assignments

$$\text{Quan} \rightarrow \mathcal{H}\text{ilb}$$

is one contemporary approach to quantum theory, *functorial quantum field theory*. By changing what we populate Quan with, replacing $\mathcal{H}\text{ilb}$ with some other class of structures, or changing what we mean by assignment, we get swaths of notions of quantum theories.

- (iii) The unitary operators of the Heisenberg picture describe reversible physical operations. By considering many particles in joint and applying a sequence of physical operations, we can build something resembling a circuit of unitaries. This is the fundamental observation and goal of *quantum computation*.

REFERENCES

- [BL11] John C. Baez and Aaron Lauda. A Prehistory of n -Categorical Physics. In Hans Halvorson, editor, *Deep Beauty: Understanding the Quantum World Through Mathematical Innovation*, pages 13–128. Cambridge University Press, Cambridge, UK, 2011.
- [Fre19] Daniel S. Freed. *Lectures on Field Theory and Topology*. Number 133 in CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, Rhode Island, 2019.
- [Ros04] Jonathan Rosenberg. A Selective History of the Stone-von Neumann Theorem. In *Operator Theory, Operator Algebras, and Applications*, volume 365 of *Contemporary Mathematics*, pages 331–354, Providence, RI, 2004. American Mathematical Society.

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⁵We call ω the frequency, and it has units of 1/time.